Reynold's Equation:

Obtain the x-velocity distribution as a function of y for simple shear flow between parallel plates.

Consider a fluid between two parallel plates. The top plate moves at a velocity $\mathbf{V_0}$ and there is a pressure drop in the **x**-direction, parallel to the plates. There is no velocity or pressure drop in the **z**-, or **y**-directions, **y** is normal to the plates. The effects of gravity are ignored and the flow is steady-state. Assume Newtonian behavior.

Under these conditions, only the x-component of the Navier-Stokes equation is needed.

$$\rho \frac{D\underline{v}}{Dt} = -\nabla P + \eta \nabla^2 \underline{v} + \rho \underline{g}$$

x − component (for Cartesian Coordinates)

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \eta \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

Steady state means that the first term (derivative in time) is 0. There is no **y** or **z** velocity so the last two terms are 0. There is not change in the **x**-velocity with **x** so the second term is 0. On the right side there is a pressure drop in **x** and the **x**-component of velocity only changes in the **y**-direction . The gravity term is dropped as it is assumed that there is no gravity effect.

For Newtonian flow $\eta\delta^2 \mathbf{v_x}/\delta \mathbf{y}^2 = \delta\tau_{\mathbf{yx}}/\delta\mathbf{y}$. The Navier-Stokes rate of change of momentum balance becomes:

$$\frac{\partial P}{\partial x} = \left(\frac{\partial \underline{\underline{\tau}}_{xy}}{\partial y} \right) = \eta \left(\frac{\partial^2 v_x}{\partial y^2} \right)$$

Of primary interest is the velocity distribution $\mathbf{v_x}$(y) that can be obtained by integration of this equation between definite limits. $\mathbf{v_x}$ at **y** = 0 is 0 and $\mathbf{v_x}$ at **y** = H, where H is the gap distance, is $\mathbf{V_0}$. Double integration yields:

$$v_x(y) = \frac{1}{\eta} \frac{\partial P}{\partial x} \left(\frac{y^2}{2} + C_1 y + C_2 \right)$$

$\mathbf{v_x}$(y=0) = 0, so C_2 = 0. $\mathbf{v_x}$(y=H) = $\mathbf{V_0}$, so $C_1 = \eta\mathbf{V_0}/(H\ \delta P/\delta x) - H/2$, then,

$$v_x(y) = \frac{1}{\eta} \frac{\partial P}{\partial x} \left(\frac{y^2}{2} + \frac{y\eta V_0}{H \partial P/\partial x} - \frac{Hy}{2} \right)$$

$$= \frac{1}{2\eta} \frac{\partial P}{\partial x} y(y - H) + V_0 \frac{y}{H}$$

The first term is quadratic with a maximum at H/2 as might be expected for a pressure driven flow. The second term is linear, reflecting a constant rate of strain for simple shear in the absence of a pressure driving force. The two components are summed. This equation is a simple form of the Reynolds equation.

The Reynold's equation for velocity is integrated in **y** to yield the integrated flow rate, $\mathbf{q_x}$,

$$q_x = \int_{y=0}^{y=H} v_x(y)dy = \left[\frac{1}{2\eta}\frac{\partial P}{\partial x}\left(\frac{y^3}{3} - \frac{Hy^2}{2}\right) + V_0\frac{y^2}{2H}\right]_{y=0}^{y=H}$$

$$= \frac{HV_0}{2} - \frac{1}{2\eta}\frac{\partial P}{\partial x}\left(\frac{H^3}{6}\right)$$

The Reynold's number for this system is given by, $\mathbf{Re} = \mathbf{H}\rho\mathbf{V_0}/\eta$, where ρ is the density. The integrated flow rate can be rewritten,

$$\rho q_x = \mathrm{Re}\left(\frac{\eta}{2} - \frac{1}{12}\frac{\partial P}{\partial x}\frac{H^2}{V_0}\right)$$

The mass flow rate depends on the Reynold's number.