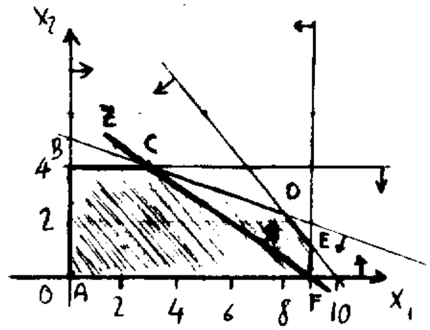
Solutions
HW. #3

3.1

$$\text{Max } Z = 12X_1 + 18X_2$$

$$\begin{aligned} \text{s.t.}: \quad 6X_1 + 5X_2 &\leq 60 \\ X_1 + 3X_2 &\leq 15 \\ X_1 &\leq 9 \\ X_2 &\leq 4 \\ X_1, X_2 &\geq 0 \end{aligned}$$



- Basis Representation for $B^A = \{S_1, S_2, S_3, S_4\}$, $A = (0,0)$

$$\begin{aligned} S_1 &= 60 - 6X_1 - 5X_2 \\ S_2 &= 15 - X_1 - 3X_2 \\ S_3 &= 9 - X_1 \\ S_4 &= 4 - X_2 \\ \hline Z &= 0 + 12X_1 + 18X_2 \end{aligned}$$

- X_2 enter the basis

$S_1 \geq 0 \iff 60 - 5X_2 \geq 0$; $X_2 \leq 12$

$S_2 \geq 0 \iff 15 - 3X_2 \geq 0$; $X_2 \leq 5$

$S_3 \geq 0 \iff \forall X_2 \in [0, \infty[$

$S_4 \geq 0 \iff 4 - X_2 \geq 0$; $X_2 \leq 4$ $\quad S_4$ leaves basis

$\Rightarrow \quad X_2 = 4 - S_4$

Basis Representation for $B^B = \{S_1, S_2, S_3, X_2\}$

$S_1 = 60 - 6X_1 - 5(4 - S_4)$	$S_1 = 40 - 6X_1 + 5S_4$
$S_2 = 15 - X_1 - 3(4 - S_4)$	$S_2 = 3 - X_1 + 3S_4$
$S_3 = 9 - X_1$	$S_3 = 9 - X_1$
$X_2 = 4 - S_4$	$X_2 = 4 - S_4$
$Z = 12X_1 + 18(4 - S_4)$	$Z = 72 + 12X_1 - 18S_4$

- x_1 enter the basis

$$S_1 \geq 0 \iff 40 - 6x_1 \geq 0 \;, \quad x_1 \leq \frac{20}{3}$$
$$S_2 \geq 0 \iff 3 - x_1 \geq 0 \;, \quad x_1 \leq 3 \qquad S_2 \text{ leaves basis}$$
$$S_3 \geq 0 \iff 9 - x_1 \geq 0 \;, \quad x_1 \leq 9$$
$$x_2 \geq 0 \iff \forall\, x_1 \in [0, \infty[$$

$$\Rightarrow \quad x_1 = 3 - S_2 + 3S_4$$

Basis Representation $B^C = \{S_1, x_1, S_3, x_2\}$

$S_1 = 40 - 6(3 - S_2 + 3S_4) + 5S_4$	$S_1 = 22 + 6S_2 - 13S_4$
$x_1 = 3 - S_2 + 3S_4$	$x_1 = 3 - S_2 + 3S_4$
$S_3 = 9 - (3 - S_2 + 3S_4)$	$S_3 = 6 + S_2 - 3S_4$
$x_2 = 4 - S_4$	$x_2 = 4 - S_4$
$Z = 72 + 12(3 - S_2 + 3S_4) - 18S_4$	$Z = 108 - 12S_2 + 18S_4$

- S_4 enter the basis

$$S_1 \geq 0 \iff 22 - 13S_4 \geq 0 \;, \quad S_4 \leq \frac{22}{13} \qquad S_1 \text{ leaves basis}$$
$$x_1 \geq 0 \iff 3 + 3S_4 \geq 0 \;, \quad \forall\, S_4 \in [0, \infty[$$
$$S_3 \geq 0 \iff 6 - 3S_4 \geq 0 \;, \quad S_4 \leq 2$$
$$x_2 \geq 0 \iff 4 - S_4 \geq 0 \;, \quad S_4 \leq 4$$

$$\Rightarrow \quad S_4 = (22 + 6S_2 - S_1)/13$$

Basis Representation for $B^0\{S_4, x_1, S_3, x_2\}$

$S_4 = (22 + 6S_2 - S_1)/13$	$S_4 = (22 + 6S_2 - S_1)/13$
$x_1 = 3 - S_2 + 3(22 + 6S_2 - S_1)/13$	$x_1 = (105 + 5S_2 - 3S_1)/13$
$S_3 = 6 + S_2 - 3(22 + 6S_2 - S_1)/13$	$S_3 = (12 - 5S_2 + 3S_1)/13$
$x_2 = 4 - (22 + 6S_2 - S_1)/13$	$x_2 = (30 - 6S_2 + S_1)/13$
$Z = 108 - 12S_2 + 18(22 + 6S_2 - S_1)/13$	$Z = (1800 - 48S_2 - 18S_1)/13$

$\Rightarrow$ STOP

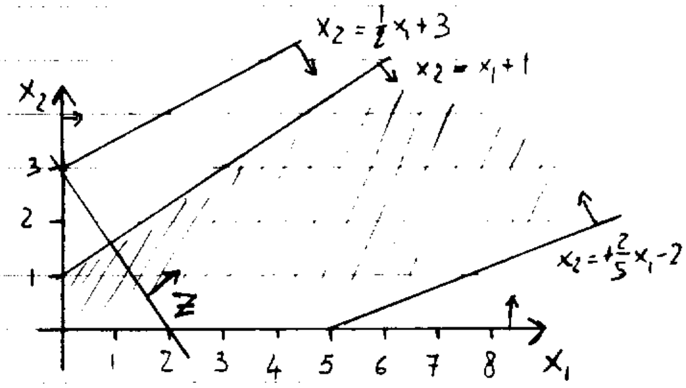
$$D \begin{cases} X_1 = \frac{105}{13} \\ X_2 = \frac{30}{13} \end{cases} \qquad Z = \frac{1800}{13}$$

Unique Optimal Solution

Solutions to HW. #3

$\boxed{3.3}$ Max $Z = 3x_1 + 2x_2$

s.t. $-x_1 + 2x_2 \le 6$
$2x_1 - 5x_2 \le 10$
$-8x_1 + 8x_2 \le 8$
$x_1, x_2 \ge 0$



* Basis Representation for $B^A = \{s_1, s_2, s_3\}$, $A = (0,0)$

$S_1 = 6 + x_1 - 2x_2$
$S_2 = 10 - 2x_1 + 5x_2$
$S_3 = 1 + x_1 - x_2$

$Z = 0 + 3x_1 + 2x_2$

- x_1 enters the basis

$S_1 \ge 0 \iff 6 + x_1 \ge 0$; $\forall x_1 \in [0, \infty[$
$S_2 \ge 0 \iff 10 - 2x_1 \ge 0$; $\boxed{x_1 \le 5}$ $\quad$ S_2 leaves basis
$S_3 \ge 0 \iff 1 + x_1 \ge 0$; $\forall x_1 \in [0, \infty[$

$$\Rightarrow \quad x_1 = \frac{5}{2}x_2 - \frac{1}{2}S_2 + 5$$

$$Z = 3\left(\frac{5}{2}x_2 - \frac{1}{2}S_2 + 5\right) + 2x_2 = 15 + \frac{19}{2}x_2 - \frac{3}{2}S_2$$

* Basis Representation for $B^B = \{S_1, x_1, S_3\}$

$S_1 = 6 + \left(\frac{5}{2}x_2 - \frac{1}{2}S_2 + 5\right) - 2x_2 = 11 + \frac{1}{2}x_2 - \frac{1}{2}S_2$
$x_1 = 5 + \frac{5}{2}x_2 - \frac{1}{2}S_2$
$S_3 = 1 + \left(\frac{5}{2}x_2 - \frac{1}{2}S_2 + 5\right) - x_2 = 6 + \frac{3}{2}x_2 - \frac{1}{2}S_2$

$Z = 15 + \frac{19}{2}x_2 - \frac{3}{2}S_2$

- x_2 enters the basis

$$s_1 \geq 0 \iff 11 + \frac{1}{2} x_2 \geq 0 \quad \forall\ x_2 \in [0, \infty[$$

$$x_1 \geq 0 \iff 5 + \frac{5}{2} x_2 \geq 0 \quad \forall\ x_2 \in [0, \infty[$$

$$s_3 \geq 0 \iff 6 + \frac{3}{2} x_2 \geq 0 \quad \forall\ x_2 \in [0, \infty[$$

$\Rightarrow$ STOP   This optimization problem is unbounded

$\boxed{3.19}$

* **Assumptions**

① There is no initial stock of forms to be reconditioned. This means that all new forms must be used on days 1, 2 and 3 because 2-day reconditioned forms will not be available until day 4.

② For the 2-day forms, forms can be taken for reconditioning at the end of the day they were used, but two full days must elapse for reconditioning, making them available on the morning of the third day after the day of their previous use

③ For the 4-day forms, these are not available until the fifth day after the day of their previous day.

④ There is no cost to hold a reconditioned form for later use.

⑤ A form can be reconditioned any number of times.

* **Variables and Parameters**

a_t = Demand for forms on day t

$X0_t$ = # New forms used on day t

$X2_t$ = # Reconditioned forms, using 2-day process, used on day t

$X4_t$ = " " " 4-day " " " " "

$Y2_t$ = # of forms taken at the end of day t for 2-day reconditioning

$Y4_t$ = # " " " " " " " " " " 4-day "

$S2_t$ = # of properly used and usable on that day forms reconditioned by the 2-day process that are in storage at the beginning of day t

$S4_t$ = Same as above for form reconditioned by 4-day process.

* **Constraints**

- Forms taken for reconditioning cannot exceed the forms used that day

$$Y2_t + Y4_t \leq a_t \qquad t = 1, \ldots, 10$$

$$Y4_6 = Y4_7 = Y4_8 = Y4_9 = Y4_{10} = 0$$
$$Y2_8 = Y2_9 = Y2_{10} = 0$$

No form shoul be taken if they cannot be used in the time frame

- Storage is calculated by accumulating the numbers that are reconditioned and subtracting the number that are then used.

2-day Process

$$S2_1 = S2_2 = S2_3 = 0$$
$$S2_4 = Y2_1$$
$$S2_5 = (S2_4 + Y2_2) - X2_4 =$$
$$= (Y2_1 + Y2_2) - X2_4$$
$$S2_6 = S2_5 + Y2_3 - X2_5$$
$$= (Y2_1 + Y2_2 + Y2_3) - (X2_4 + X2_5)$$

In general $\quad S2_t = \sum_{k=1}^{t-3} Y2_k - \sum_{k=4}^{t-1} X2_k \qquad t = 5, 6 \ldots, 10$

4-day Process

$$S4_1 = S4_2 = S4_3 = S4_4 = S4_5 = 0$$
$$S4_6 = Y4_1$$
$$S4_7 = S4_6 + Y4_2 - X4_6$$

In general $\quad S4_t = \sum_{k=1}^{t-5} Y4_k - \sum_{k=6}^{t-1} X4_k \qquad t = 7, 8, 9, 10$

- The demand is equal to the forms used

$$XO_1 = a_1$$
$$XO_2 = a_2$$
$$XO_3 = a_3$$
$$XO_4 + X2_4 = a_4 \quad \text{(now 2-day forms are available)}$$
$$XO_5 + X2_5 = a_5$$
$$XO_6 + X2_6 + X4_6 = a_6 \quad \text{(now 4-day forms are available)}$$
$$XO_t + X2_t + X4_t = a_t \quad t = 7, 8, 9, 10$$

$$X2_1 = X2_2 = X2_3 = 0$$
$$X4_1 = X4_2 = X4_3 = X4_4 = X4_5 = 0$$

- Positivity of all variables

$$XO_t ; X2_t ; X4_t ; Y2_t ; Y4_t ; S2_t ; S4_t \geq 0 \qquad t = 1, 2, \ldots, 10$$

* Objective ⇒ Minimize Costs

$$\text{Min } Z = 200 \sum_{t=1}^{10} XO_t + 75 \sum_{t=4}^{10} X2_t + 25 \sum_{t=6}^{10} X4_t$$