LP Sensitivity Analysis.

There are two types of LP sensitivity Analysis: "Right hand Side" (RHS) & Objective Sens. Anal.

RHS Sensitivity Analysis

The "right hand side" is the coefficient on the right hand side of a constraint function.

We wish to know two things: ① What would be the change in the optimal objective function value, Z^*, given a unit increase in the RHS coefficient value, and ② over what range of RHS values this change is valid.

Mathematically, we are calculating the derivative:

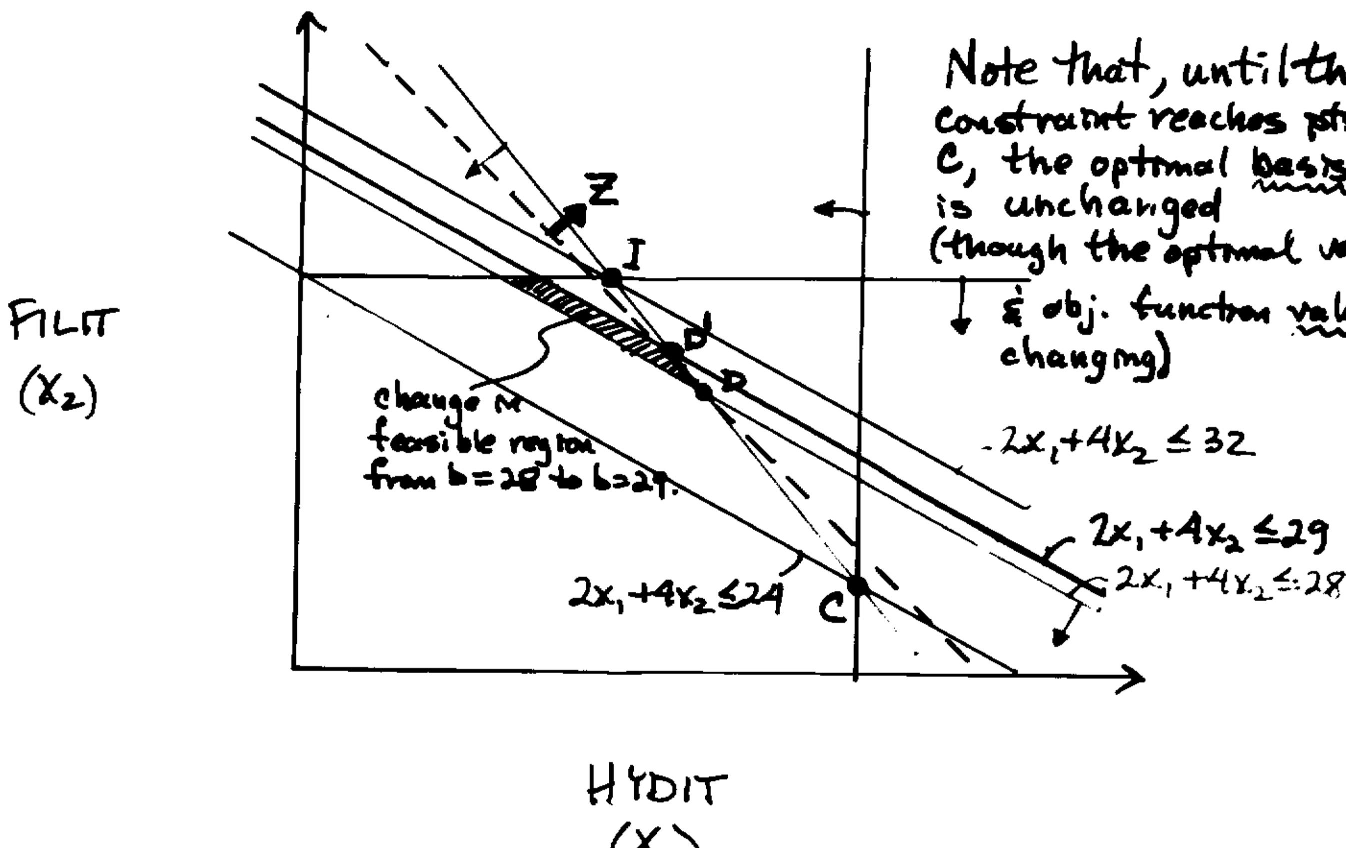
$$\left.\frac{dZ^*}{db}\right|_{b=b_0} \qquad \text{where } b = \text{RHS coefficient}$$

and the range of validity of this derivative:

$$b_0 - \delta b_1 \leq b \leq b_0 + \delta b_2$$

GRAPHICAL INTERPRETATION OF RHS Sensitivity Analysis.

Recall the Homewood Masonry Example



Note that, until the constraint reaches pts I or C, the optimal basis is unchanged (though the optimal variable & obj. function values are changing)

$Z^*(b=28) = 1480$
$Z^*(b=29) = 1490$
So Z^* increases by $10 for each unit increase in b (wabash red clay availability)

RHS Sensitivity Information

$$\left.\frac{dZ^*}{db}\right|_{b=28} = 10 \left[\frac{\$ \text{ PROFIT}}{m^3 \text{ Wabash clay}}\right] \Leftarrow \text{"Shadow Price" -OR- "Dual Price"}$$

Allowable increase in b = 4 m^3
Allowable decrease in b = 4 m^3

Obtaining RHS Sensitivity Information From the optimal basis representation.

Recall the optimal basis for the H.M. Problem:

$B^D = \{S_3, X_2, X_1, S_4\}$

$$
\boxed{
\begin{array}{l}
S_3 = 2 - \frac{1}{2}S_1 + \frac{3}{5}S_2 \\
X_2 = 4 - \frac{1}{2}S_1 + \frac{1}{5}S_2 \\
X_1 = 6 + \frac{1}{2}S_1 - \frac{2}{5}S_2 \\
S_4 = 2 + \frac{1}{2}S_1 - \frac{1}{5}S_2 \\
\hline
Z = 1480 - 10S_1 - 24S_2
\end{array}
}
$$

The slack variable S_1 corresponds to the red clay constraint:

$$2x_1 + 4x_2 + S_1 = 28$$

-OR-

$$2x_1 + 4x_2 = (28 - S_1)$$

Q: How does increasing S_1 correspond to a change in the RHS ($b = 28$) of this constraint?

A: A unit increase in $S_1 \rightarrow$ a unit decrease of b.

Q: How does a unit decrease in b affect the optimal objective function value Z^*?

A: It decreases Z^* by \$10 $\rightarrow$ this is obtained directly from the reduced costs associated with the slack variable for the associated constraint.

Q: How much can b be decreased before this shadow price ($= 10\ \$/m^3$) is no longer valid?

A: Until a basis change would occur → corresponding to pt. C in Diagram of feasible region.

Critical Constraint:

$$S_3 = 2 - \frac{1}{2}S_1 + \frac{2}{5}S_2$$

S_1 increases by 4, S_3 goes to zero } New basis @ pt. C.

* Thus S_1 could be increased by 4 units, or b decreased by 4 units, and the sensitivity analysis ($dz^*/db = 10\ \$/m^3$) is still valid.

SIMILAR STATEMENTS HOLD TRUE FOR A DECREASE IN S_1 → How much can we decrease S_1 (or, increase b)?

CRITICAL CONSTRAINT:

$$S_4 = 2 + \frac{1}{2}S_1 - \frac{1}{5}S_2$$

New basis @ pt. I { S_1 decreases by 4 ($S_1 = -4$, or $b = 28 - (-4) = 32$), S_4 goes to zero

* Thus S_1 could be decreased by 4 units, or b increased by 4 units, and the sensitivity analysis ($dz^*/db = 10\ \$/m^3$) is still valid.

We would summarize as follows:

Constraint	Current RHS	Optimal Usage	Allowable Decrease	Allowable Increase	Shadow Price
Red Clay Resource	$28\ m^3$	$28\ m^3$	$4\ m^3$	$4\ m^3$	$10\ \$/m^3$

Q: What would sensitivity analysis information look like for non-binding constraints?

Example (a minimization problem - optimal Basis).

$B^* = \{x_3, x_1\}$

$$x_3 = \frac{1}{3} - \frac{1}{3}x_2 - \frac{1}{3}S_1 + \frac{2}{3}S_2$$

$$x_1 = \frac{5}{3} - \frac{2}{3}x_2 + \frac{1}{3}S_1 + \frac{1}{3}S_2$$

$$Z = \frac{11}{3} + \frac{4}{3}x_2 + \frac{1}{3}S_1 + \frac{4}{3}S_2$$

ORIGINAL PROBLEM:

Minimize $Z = 2x_1 + 3x_2 + x_3$

s.t. $2x_1 + x_2 - x_3 - S_1 = 3$

$x_1 + x_2 + x_3 - S_2 = 2$

$x_1, x_2, x_3, S_1, S_2 \geq 0$

Q: Develop sensitivity information for both constraints from the optimal basis. Express in tabular form as above.

A: Constraint 1

Increasing S_1 by one unit increases Z^* by $\boxed{1/3}$ objective function unit. This corresponds to the effect of increasing the rhs of constraint 1 by one unit.

Allowable increase in S_1 (and rhs value) $\boxed{+1}$ unit (from 1st constraint)

Allowable decrease in S_1 (and rhs value) $\boxed{-5}$ units (from 2nd constraint)

Summary

Constraint	Current RHS	Opt Usage	Allow. Decrease	Allow. Increase	Shadow Price
1	3	3	5	1	1/3
2	2	2	1/2	∞	4/3

Q: What if x_1, x_2, x_3 were required to be ≥ 1 instead of zero?

$x_1 \geq 0$	0	5/3	∞	5/3	0
$x_2 \geq 0$	0	0	—	1	4/3
$x_3 \geq 0$	0	1/3	∞	1/3	0

- Within these limits, since x^* remains unchanged, it is fairly easy to calculate the effect of a unit change in a coefficient:

$$Z = c_1 x_1^* + c_2 x_2^*$$

$$\partial Z^* / \partial c_1 = x_1^* \quad \partial Z^* / \partial c_2 = x_2^*$$

So we might summarize the Obj. function sensitivity analysis in a tabular form as follows, one row for each structural variable.

Structural Variable	Optimal Value	Current Coefficient	Allowable Decrease	Allowable Increase	Unit Change In Profit
x_1	6	140	60	20	6
x_2	4	160	20	120	4

The "allowable decrease/increase" are calculated by first calculating the basis change for all adjacent extreme points ('C' & 'E') — this is the usual procedure, except that we are no longer concerned with optimality — and then requiring that the change in the objective function which would result is zero:

$$\begin{pmatrix} \text{Optimal Obj.} \\ \text{function value} \\ \text{@ original} \\ \text{optimal basis} \end{pmatrix} = \begin{pmatrix} \text{Optimal Obj.} \\ \text{function value} \\ \text{@ adjacent} \\ \text{extreme pt.} \end{pmatrix}$$

Ex.

Recall the optimal basis for H.M. Problem:

$$S_3 = 2 - \tfrac{1}{2} S_1 + \tfrac{2}{5} S_2$$

optimal structural variable values.

$$X_2 = 4 - \tfrac{1}{2} S_1 + \tfrac{1}{5} S_2$$

$$X_1 = 6 + \tfrac{1}{2} S_1 - \tfrac{2}{5} S_2$$

$$S_4 = 2 + \tfrac{1}{2} S_1 - \tfrac{1}{5} S_2$$

$$Z = 1480 - 10 S_1 - 24 S_2$$

If S_1 were to Increase from zero and become basic:

$S_1 = 4$, S_3 becomes non-basic (which adjacent vertex is this in figure?)

and the new values of X_1, X_2 would become:

$X_2 = 2 \quad X_1 = 8$

And so the condition of equal objective function values at adjacent extreme pts. yields:

$$\underbrace{c_1 \cdot 6 + c_2 \cdot 4}_{\text{optimal basis (original)}} = \underbrace{c_1 \cdot 8 + c_2 \cdot 2}_{\text{adjacent basis}} \Leftarrow$$

Requires no change in Z if basis change is made.

If $C_1 = 140$, then solving for C_2,

$$2C_2 = 2C_1 = 2(140); \quad C_2 = \frac{280}{2} = \boxed{140}$$

Allow. Decrease = 20

If $C_2 = 160$, solving for C_1,

$$2C_1 = 2C_2 = 2(160); \quad C_1 = \frac{320}{2} = \boxed{160}$$

Allow. Increase = 20

* As mentioned by Revelle et. al., this procedure requires some modification in higher dimensions, but remains basically the same.