Multiobjective Analysis.

Thus far, we have considered problems in which a single objective function is minimized or maximized.

In fact, most real-world problems are more naturally expressed using multiple objectives. This module is concerned with methods of analysis for such problems.

It should not surprise us that many problems are multiple-objective. Think of the multiplicity of uses of the most common of civil works - a building?!

Some specific examples:

Design & operation of surface water reservoirs.

Objective	Math. Approximation
- Water supply	(maximize water stored)
- Flood Control	(minimize water stored)
- Hydroelectric gen.	(maximize water volume released + hydraulic head)
- In stream flow for habitat	(mimic as closely as possible natural flow + variation)
- Recreation	(minimize variation about a target elevation).

Design & Operation of "Finished water" Storage tanks.

Objective	Math. Approximation
- Water supply	(maximize water stored)
- Energy efficiency	(minimize water pumping peak rate + duration)
- Water Quality	(minimize average residence time of stored water).

Think about how many times you failed to see a problem from another persons "point of view". We would say that you were missing one or more of that persons key objectives (or that you had a different emphasis). This is common!

A motivating Example

Homewood masonry is now concerned about the environmental effects of their production processes.

Lab. tests show that particulates released during blending can be hazardous & are released to the atmosphere. Further, approximately 500 mg. particulates are released per ton HYDIT, & 200 mg per ton FILIT.

$$\text{Minimize } Z_2 = 500x_1 + 200x_2$$

(An environmental objective to compete with cost)

Note:
- The units of environmental quality objective (mg Particulates) are incommensurate with those for Profit (\$). (i.e. they are not comparable directly).
- Env. Quality appears to conflict with Profit goals (what solution maximizes env. quality, or min. particulate release?)

The multiobjective problem:

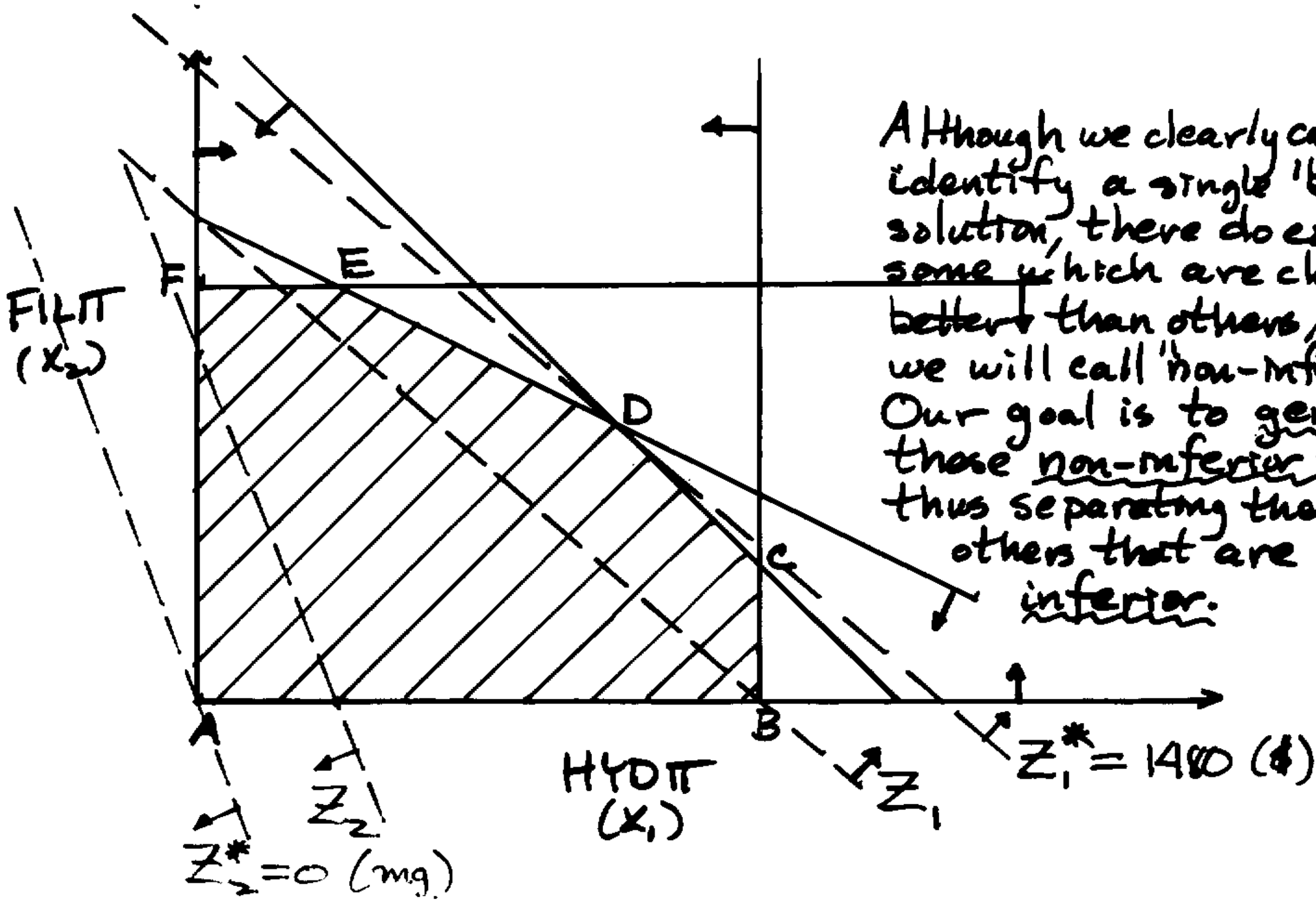
$$\begin{aligned}
\text{Maximize } Z_1 &= 140x_1 + 160x_2 \\
\text{Minimize } Z_2 &= 500x_1 + 200x_2
\end{aligned}$$

Two objectives - what to do? (what is an "optimal solution" in this case?)

$$\begin{aligned}
\text{s.t.} \quad 2x_1 + 4x_2 &\leq 28 \\
5x_1 + 5x_2 &\leq 50 \\
x_1 &\leq 8 \\
x_2 &\leq 6 \\
x_1, x_2 &\geq 0
\end{aligned}$$

Answer: There is no one "optimal solution" to a multi-objective problem — only a <u>set</u> of alternative solutions called the "non-inferior set".

Recall the feasible region & Objective function contours (now adding those for Z_2):



Although we clearly can not identify a single 'best' solution, there do exist <u>some</u> which are clearly <u>better</u> than others, which we will call "non-inferior". Our goal is to <u>generate</u> those <u>non-inferior solutions</u>, thus separating them from others that are <u>inferior</u>.

	Alternative	Z_1 (max)	Z_2 (min)	Noninferiority
Z_2 Single-Obj. Optimum →	(A)	0	0	Noninferior
	B	1120	4000	<u>Inferior</u> (Dominated by D,E)
	C	1440	4400	<u>Inferior</u> (Dominated by D)
Z_1 Single-Objective Optimum →	(D)	1480	3800	Noninferior
	E	1240	2200	Noninferior
	F	960	1200	Noninferior.

The vertices A, D, E, F are part of the <u>non-inferior set</u> (there are other solutions as well, as we will see). The vertices B, C are not, and would not be part of any solution to this problem. It is impossible to say which of A, D, E, F should be implemented w/o <u>eliciting Preferences</u>.

The job of the systems analyst (you) is to generate the non-inferior set (or, an approximation of it) and present it to the decision maker.

The noninferior set is usually presented as a tradeoff curve plotted in "objective space".

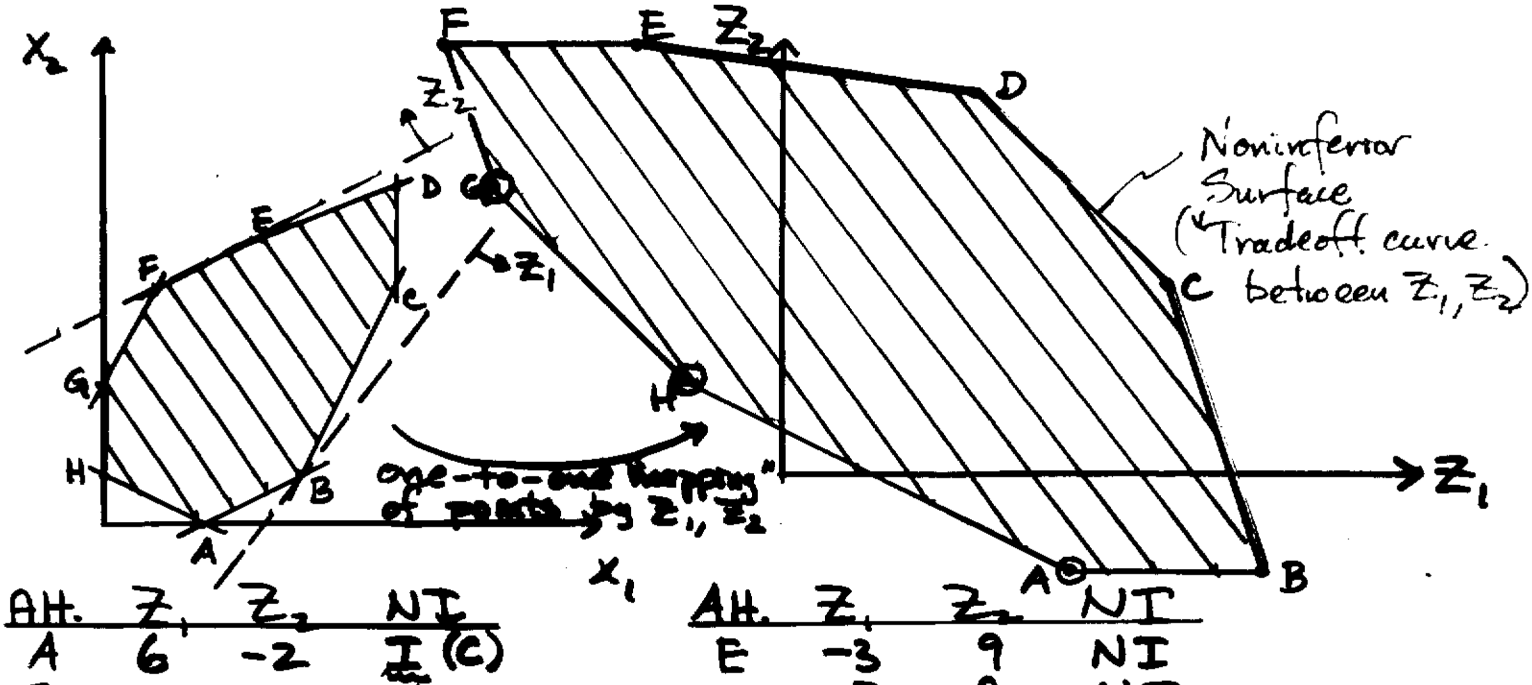
Consider the more complex 2-objective problem given as section 4.1.3 in Rwo:

$$\text{Maximize} \begin{Bmatrix} Z_1 = 3x_1 - 2x_2 \\ Z_2 = -x_1 + 2x_2 \end{Bmatrix}$$

Subject to:

$$\begin{aligned} 4x_1 + 8x_2 &\geq 8 \\ 3x_1 - 6x_2 &\leq 6 \\ 4x_1 - 2x_2 &\leq 14 \\ x_1 &\leq 6 \\ -x_1 + 3x_2 &\leq 15 \\ -2x_1 + 4x_2 &\leq 18 \\ -6x_1 + 3x_2 &\leq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Decision Space — Objective Space



Alt.	Z_1	Z_2	NI
A	6	-2	I (C)
B	10	-2	NI
C	8	4	NI
D	4	8	NI

Alt.	Z_1	Z_2	NI
E	-3	9	NI
F	-7	9	NI
G	-6	6	I (E,D)
H	-2	2	I (D,C)

How to generate the Non-Inferior Set of Solutions. (leaving example problem for the moment)

A. Weighting Method.

A.I. Form a 'composite' Objective function (the 'Grand Objective')

$$\boxed{Z_G = w_1 Z_1 + w_2 Z_2}$$

where (w_1, w_2) are positive, constant, weights that express the relative emphasis on the two objectives.

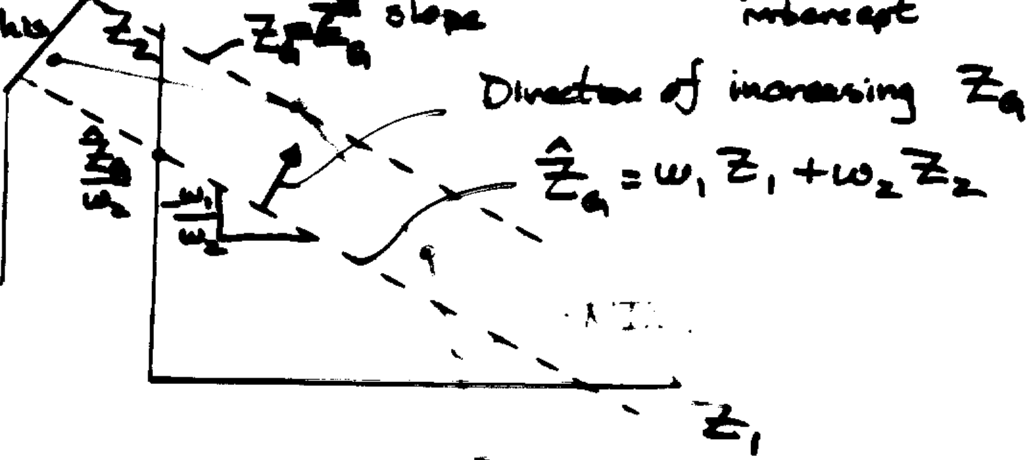
What does the grand objective look like when viewed in objective space?

Plotting a line with constant $Z_G = \hat{Z}_G$

$$\hat{Z}_G = w_1 Z_1 + w_2 Z_2$$

$$Z_2 = \underbrace{-\left(\frac{w_1}{w_2}\right)}_{\text{slope}} Z_1 + \underbrace{\frac{\hat{Z}_G}{w_2}}_{\text{intercept}}$$

Note: Maximizing Z_G yields a Non-inf. Solution corresponding to a vertex of the problem. This non-inf. soln. depends only on the relative values of the weights w_1/w_2.



A.II. Solve a series of p different optimization problems where the objective is the grand objective Z_G and each problem only differs in the values of the weights w_1, w_2.

$$\underset{\underline{x}}{\text{Max}} \; Z_G = \underbrace{w_1 Z_1 + w_2 Z_2}$$

assumes Z_1, Z_2 are to be maximized.

subject to: $\underbrace{\underline{x} \in F_d}$

This is shorthand for the constraint set of the Multi-objective problem.

w_1	w_2	w_1 / w_2
0.0	1.0	0
0.2	0.8	0.25
0.4	0.6	0.33
0.6	0.4	1.5
0.8	0.2	4.0
1.0	0.0	∞

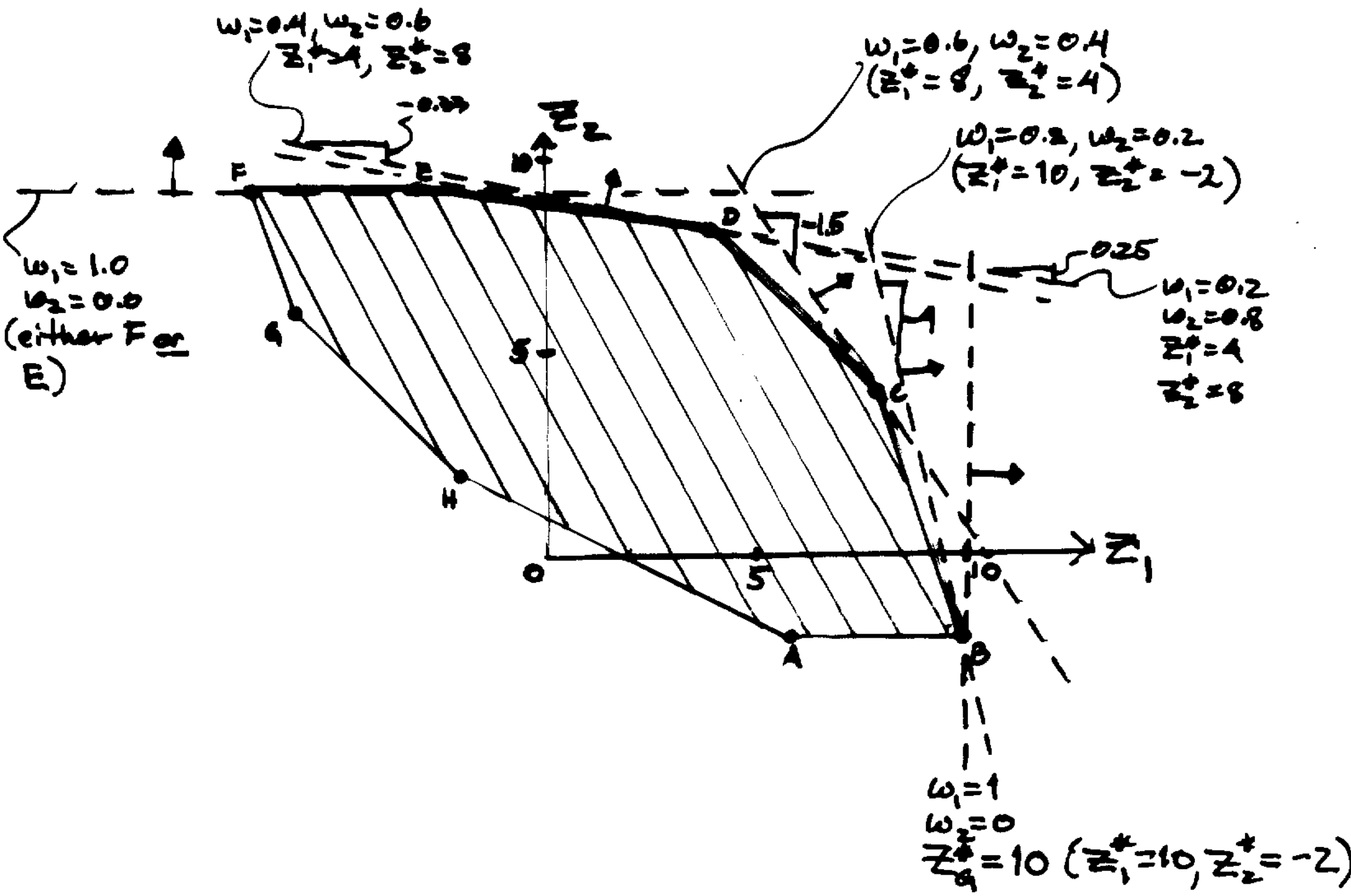
} $p=6$ (6 diff. optimization problems)

You can choose how finely to vary the weights. A finer variation results in a more accurate approximation of the noninferior set.

Going back to example problem...

$$Z_G = \omega_1(3x_1 - 2x_2) + \omega_2(-x_1 + 2x_2)$$
$$= (3\omega_1 - \omega_2)x_1 + (-2\omega_1 + 2\omega_2)x_2$$

We will maximize this Grand Objective. In objective space this looks like:



B. The Constraint Method

B.1. Solve an Augmented problem where we maximize Z_1 subject to the original constraints, plus new constraints requiring a minimum level of each other objective

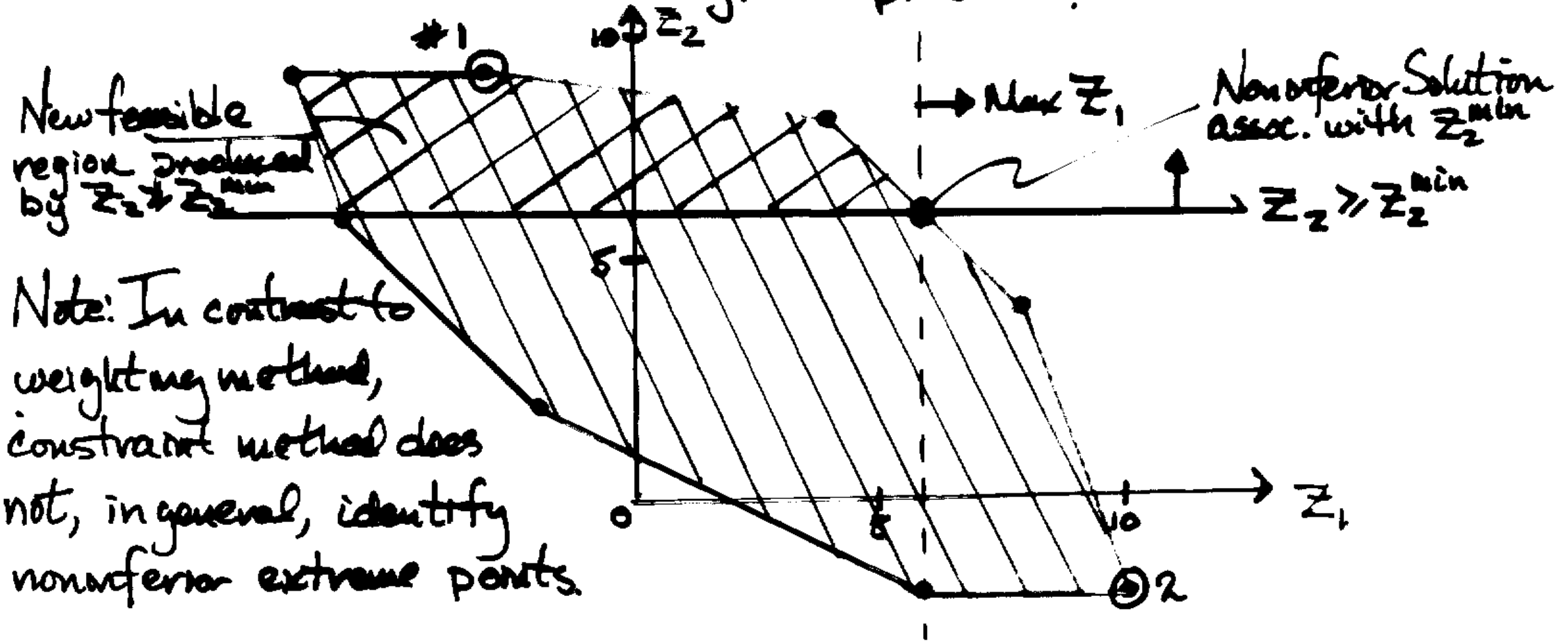
$$\max_{\underline{x}} \; Z_1$$

subject to:

$$\underline{x} \in F_D$$

$$\left.\begin{array}{l} Z_2 \geq Z_2^{min} \\ \vdots \\ Z_m \geq Z_m^{min} \end{array}\right\} \text{Additional Constraints requiring min. levels of each objective.}$$

What does this do to original problem?



Note: In contrast to weighting method, constraint method does not, in general, identify noninferior extreme points.

B.2

How to vary Z_2^{min} ?

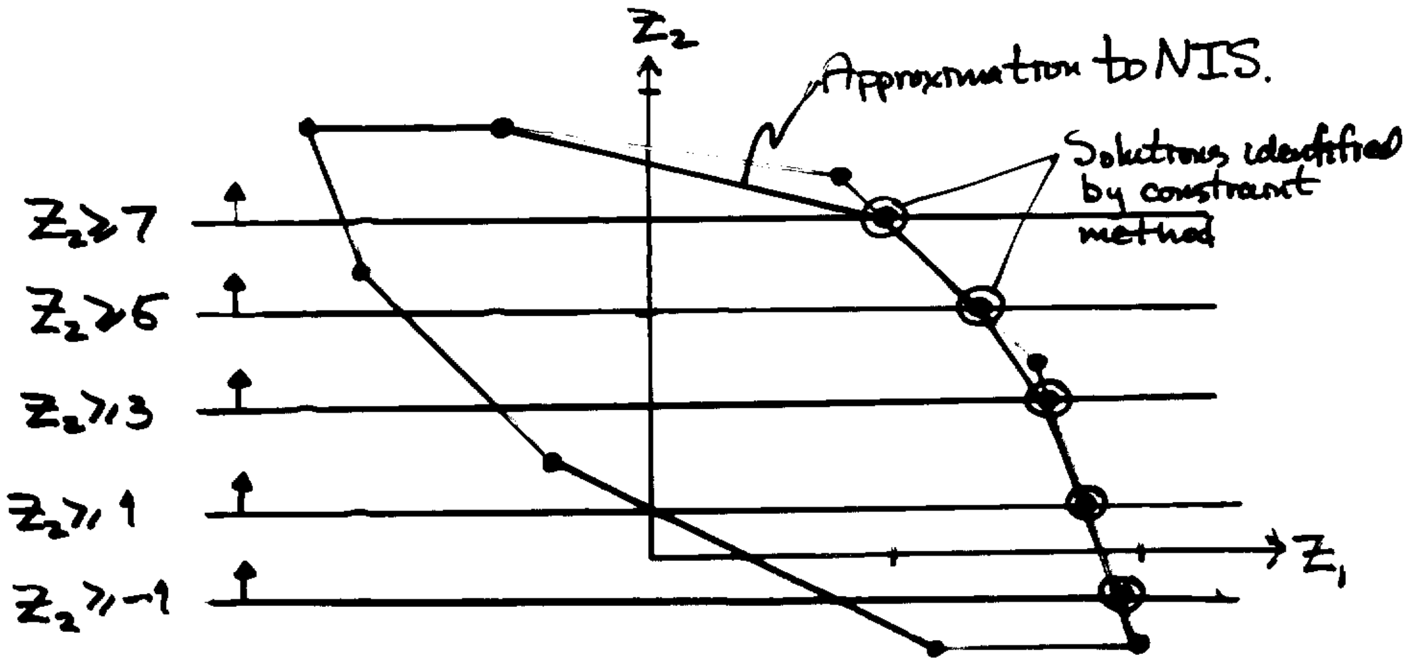
Solve two Problems:

Max Z_1 s.t. $\underline{x} \in F_D$ ⇒ (#2) pg. 8	Max Z_2 s.t. $\underline{x} \in F_D$ ⇒ (#1) pg. 8

These problems provide the range of variation of Z_2 within the noninferior set.

Select Z_2^{min} evenly spaced within this range.

In the example above, Z_2 ranges from $-2 \le Z_2 \le 9$ within the NIS. We might select $Z_2^{min} = \{-1, 1, 3, 5, 7\}$, for example.



Z2
Z1
Approximation to NIS.
Solutions identified by constraint method
Z2≥7
Z2≥6
Z2≥3
Z2≥1
Z2≥-1