Example - Multiobjective Analysis (Non-inferiority and constraint method)

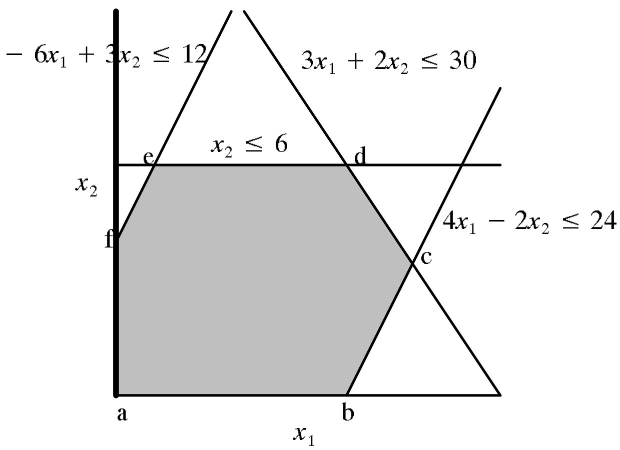
Consider the following multiobjective programming problem:

$$\max \ Z_1 = 9x_1 + 6x_2 \quad (1)$$
$$\min \ Z_2 = 3x_1 - 2x_2$$

Subject to:

$$3x_1 + 2x_2 \leq 30 \quad (2)$$
$$-6x_1 + 3x_2 \leq 12 \quad (3)$$
$$4x_1 - 2x_2 \leq 24 \quad (4)$$
$$x_2 \leq 6 \quad (5)$$
$$x_1, x_2 \geq 0 \quad (6)$$

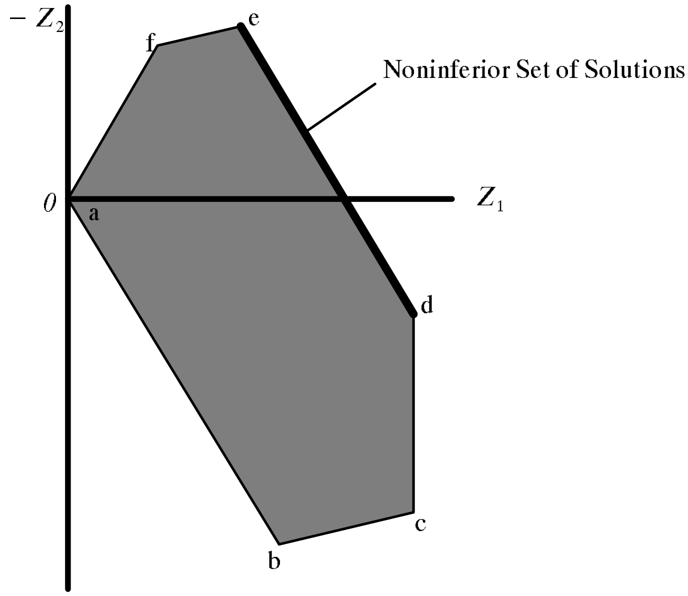
The feasible region in decision space looks like:



Below is a tabulation, for each extreme point, of the objective function values Z_1, Z_2, and a designation of whether or not the extreme point is inferior or non-inferior. For each inferior extreme point, the point(s) which dominate it are listed. You should make certain that you know how this table was generated, and in particular why the points are labelled non-inferior/inferior. Remember that a point is noninferior *if, and only if,* there does *not* exist even one other solution (extreme point) which is better in terms of *all* objective function values. Conversely, an inferior solution is one for which at least one other solution dominates it - meaning that it is better in terms of all objective function values.

Alternative	X1	X2	Z1 (max)	Z2 (min)	Noninferiority
a	0	0	0	0	Inferior (e,f)
b	6	0	54	18	Inferior (c,d)
c	7.71	3.43	90	16.27	Inferior (d)
d	6	6	90	6	Noninferior
e	1	6	45	-9	Noninferior
f	0	4	24	-8	Inferior (e)

The feasible region in objective space is obtained by plotting the Z_1, Z_2 values from the above table. In this case, I will plot $-Z_2$ versus Z_1, because we wish to minimize Z_2, and it is most convenient to view the objective space from the perspective of maximization (that way, moving to the right, or up, improves the objective function, and thus the “northeast corner” rule applies).



Using the constraint method to approximate the noninferior set - or the "tradeoff curve" - we modify the original problem by including all but one of the objectives (it doesn't matter which one) as constraints. It is easiest to think about this when all objectives are to be maximized, so you might want to first convert all problems into maximization problems, for all objectives.

The original problem is modified by placing a constraint on $-Z_2$ — what sense should be the inequality? Since this is an objective we wish to maximize, then we should include a constraint that requires *at least* a certain amount of $-Z_2$.

$$\max\ Z_1 = 9x_1 + 6x_2 \tag{7}$$

Subject to:

$$-Z_2 = -3x_1 + 2x_2 \geq Z_2^{\min}$$

$$3x_1 + 2x_2 \leq 30 \tag{8}$$

$$-6x_1 + 3x_2 \leq 12 \tag{9}$$

$$4x_1 - 2x_2 \leq 24 \tag{10}$$

$$x_2 \leq 6 \tag{11}$$

$$x_1, x_2 \geq 0 \tag{12}$$

where $Z_2^{\min}$ is a constaint right hand side value for the new objective function constraint. How big or small should this right hand side be? We can easily see from the objective space diagram that the value of $-Z_2$ in the noninferior set varies from -6 to +9, so these are our bounds. But what if we didn't have the diagram or the table?

In general, what we will do is to first solve p maximization problems, corresponding to the p original maximization objectives (p is 2 in the present case). So, for our problem, we solve one problem to first maximize objective 1, and another, separate, problem, to maximize the negative of objective 2. These two problems will identify points c, d, and e (convince yourself of this from looking at the objective space diagram). Then we construct a payoff table:

X1	X2	Z1	-Z2	Extreme Point	How Generated
7.71	3.43	90	-16.27	C	Max Z1
6	6	90	-6	D	Max Z1
1	6	45	9	E	Max -Z2

From this table, which can be generated no matter how many objectives you have, and which is the first step in any actual numerical analysis, you can see the range of variation of each of the p = 2 objectives: $45 \leq Z_1 \leq 90$ and $-6 \leq -Z_2 \leq 9$. Note that I have already excluded point C as being inferior, in my decision on the range for $-Z_2$. How do I know to do this? First, I know that there are alternate optima for the problem that maximizes objective 1, from the optimal basis (or, from the output of a commercial simplex algorithm). So I know to be suspicious of these solutions, and that in general only one of them will be noninferior (see the objective space diagram). I can either check each alternate optimal solution for noninferiority (like we did for the first table above), or I can use a little "trick" which is to resolve the maximize Z_1 problem again, but this time adding just a very small amount of objective 2, so the objective would be:

$$\max \ Z = Z_1 + \epsilon(-Z_2) \tag{13}$$

where ϵ is a very small number (like, 0.0001, for example). The idea is that ϵ is small enough to not really matter, but it is enough to cause point C to not be selected as an alternate optimal solution, and thus to expose it as being inferior. This may be a difficult concept to grasp at first. Maybe it is easier to look at the above payoff table, and think that, if you were maximizing Z_1, then you would be indifferent to either C or D (and hence they are alternative optimal solutions), but if you were maximizing Z_1 but with just a little of $-Z_2$ added, then point D would look so much better (and thus it would be selected). This is one practical way to handle the situation of alternate optimal solutions to the p maximization problems, but I shouldn't dwell to much on it because the situation doesn't really occur that frequently.

So, now, we solve different maximization problems to solve eqs. 7-12, with the values of Z_2^{min} varying from -6 to 9. This will generate an approximation to the noninferior set of solutions, and the quality of the approximation will depend on how finely you vary Z_2^{min} (In general - in this particular case the entire noninferior surface is comprised of one pair of noninferior vertices and their connecting line, and so we can generate the exact noninferior set with only two points. But this is not typical).

Graphically, we can show the solutions generated if we vary Z_2^{min} in increments of 3. The large dots indicate the actual solutions obtained by the constraint method; coupled with the noninferior solutions d and e, they can be used to approximate the noninferior set. Practically, we would plot these solutions on an objective space graph - without plotting the entire feasible region as we have done for the example problems - and just connect them with lines. In this case the approximation is exact, but in general the true noninferior set could lie some distance to the northeast of the approximation, between the actual points that are plotted.