Exponential Integrals (From Ed DiMarzio NIST/NIH)

$$G(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

1) Square the function in 2 Cartesian dimensions

$$(G(\alpha))^2 = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-\alpha (x^2+y^2)}$$

2) Convert to cylindrical coordinates

$$(G(\alpha))^2 = \int_0^{\infty} \int_0^{2\pi} e^{-\alpha r^2} r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r\,dr\, e^{-\alpha r^2}$$

$$= \frac{-2\pi}{2\alpha} \int_0^{\infty} -2\alpha r\,dr\, e^{-\alpha r^2}$$

$$(G(\alpha))^2 = \frac{-\pi}{\alpha} \left[e^{-\alpha r^2} \right]_0^{\infty} = \frac{\pi}{\alpha}$$

so

$$G(\alpha) = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

<u>Higher Power Even Moments $H(\alpha)$</u>

$$H(\alpha) = \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx$$

$$H(\alpha) = \frac{d\,G(\alpha)}{d\alpha} \quad \text{We know } G(\alpha) = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$\text{so } \; H(\alpha) = \frac{1}{2} \frac{\pi^{1/2}}{\alpha^{3/2}}$$

$$U(\alpha) = \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx$$

$$= \frac{-dH(\alpha)}{d\alpha} = \frac{\pi^{1/2}}{2} d\alpha^{-3/2} = \frac{3}{4} \pi^{1/2} \alpha^{-5/2}$$

Odd moments = 0 due to symmetry