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The Numerical Calculation of Storage and Loss Compliance from Creep Data for Linear Viscoelastic Materials*)

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With 11 figures and 7 tables

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1. Introduction

One of the problems frequently occurring in the investigation of the relaxation behaviour of linear viscoelastic materials is that of converting results of dynamic measurements into the result of a creep experiment, or vice versa. Formally, these problems were solved through the theory of linear viscoelastic behaviour. According to this theory, the interconversion of various characteristic functions may be performed by application of linear integral transformations. For instance, the calculation of loss and storage compliance from the creep compliance may be performed by the *Fourier* sine and cosine transformation. However, as has been shown elsewhere (1), the actual application of those integral transformations to experimental data gives rise to basic difficulties, and to tedious calculations besides. It would therefore be highly desirable to have simple numerical formulae for those interconversions which do not involve any integration of the function measured. As far as calculation of transient response from dynamic response is concerned, numerical formulae of this type have already been given and discussed (1). It is the purpose of the present paper to discuss the inverse problem, viz. that of numerical calculation of dynamic response from creep response.

We will base our discussions on a material which obeys *Boltzmann*'s superposition principle and has a positive retardation spectrum. Under these conditions, the result of a creep experiment may be described by the creep compliance, $J(t)$, as a function of time t; it is defined as the strain as a function of time produced by a unit step in stress at time zero. The creep compliance may then be written as an integral (1):

$$J(t) = J_0 + \int_0^\infty f(\tau)\,[1 - e^{-t/\tau}]\,d\tau + t/\eta \qquad [1]$$

where J_0 is the instantaneous compliance, η the viscosity and $f(\tau)$ a non-negative function of τ, the retardation spectrum.

The result of forced vibration experiments may be described by the storage compliance, $J'(\omega)$, and the loss compliance, $J''(\omega)$, as functions of the angular frequency, ω. The definition of these quantities is based on the steady state response to a harmonic stress with unit stress amplitude and with frequency $\nu = \omega/2\pi$. Then the strain will consist of two harmonic components, one in phase with the stress and one lagging behind 90 degrees with it. The amplitudes of these components are, respectively, $J'(\omega)$ and $J''(\omega)$. The integral representations for $J'(\omega)$ and $J''(\omega)$ are (1):

$$J'(\omega) = J_0 + \int_0^\infty f(\tau)\,\frac{1}{1+\omega^2\tau^2}\,d\tau \qquad [2]$$

$$J''(\omega) = \int_0^\infty f(\tau)\,\frac{\omega\tau}{1+\omega^2\tau^2}\,d\tau + 1/\omega\eta\,. \qquad [3]$$

For later use we mention the following formula for the finite difference of the creep compliance, which is easily found from eq. [1]:

$$J(2\alpha t) - J(\alpha t) = \int_0^\infty f(\tau)\,e^{-\alpha t/\tau}\,[1 - e^{-\alpha t/\tau}]\,d\tau + \alpha t/\eta\,. \qquad [4]$$

All four expressions [1] to [4] are similar; they contain an integral over the retardation spectrum times a function of either $\omega\tau$ or t/τ which will be called the *intensity function* of the corresponding expression. If we introduce abbreviations $x = t/\tau$ and $x = 1/\omega\tau$, the intensity functions of expressions [1], [2], [3] and [4] become simply:

$$\chi(x) = 1 - \exp(-x) \qquad [5]$$

$$\chi'(x) = x^2/(1+x^2) \qquad [6]$$

$$\chi''(x) = x/(1+x^2) = \chi'(x)/x \qquad [7]$$

$$\psi(x;\alpha) = [1 - \exp(-\alpha x)]\exp(-\alpha x)\,. \qquad [8]$$

*) Dedicated to Prof. Dr. *J. Meixner*, Aachen, on the occasion of his 60th birthday.

In order to find an approximation for $J''(\omega)$ in terms of the finite difference $J(2\alpha t) - J(\alpha t)$, we will approximate the intensity function $\chi''(x)$ of $J''(\omega)$ in terms of the intensity function $\psi(x;\alpha)$ of $J(2\alpha t) - J(\alpha t)$. Similarly, to find an approximation for $J'(\omega)$ in terms of $J(t)$ and finite differences of the type $J(2\alpha t) - J(\alpha t)$, we will approximate the intensity function $\chi'(x)$ in terms of the intensity functions $\chi(x)$ and $\psi(x;\alpha)$. Once these problems have been solved appropriately, it will be found that also the terms outside the integrals in expressions [1], [2], [3] and [4] are automatically accounted for. In the resulting expressions for the approximation formulae, neither the instantaneous compliance, J_0, nor the viscosity, η, will occur explicitly.

2. Numerical Formulae for Calculation of Storage Compliance from Creep Compliance

Various numerical formulae for the calculation of $J'(\omega)$ from $J(t)$ are listed in table 1. All these formulae have one feature in common. The calculation is based on values of the creep compliance at times that are equally spaced on a logarithmic time scale. The ratio between these succeeding times corresponds to a factor of two. This type of sampling of the creep compliance had been chosen with regard to the technique of creep measurement employed at our institute (2). Using a logarithmic clock (3), the digital registration unit of the creep apparatus is activated at the above mentioned logarithmic sequence of times, viz. 2 seconds, 4 sec, 8 sec, 16 sec, etc. after the start of the creep experiment. Therefore, the item of information needed before those formulae can be applied, is just the one obtained by the digital creep technique.

For a discussion we select two formulae, viz. the simplest and the most involved formula of table 1. The simple formula is:

$$J'(\omega) \sim J(t) - 0.86\,[J(2t) - J(t)] = 1.86\,J(t) - 0.86\,J(2t) \qquad [9]$$

where

$$\omega = 2\pi\nu = 1/t\,.$$

Notwithstanding its simplicity, this formula may be quite useful in a number of cases; if the damping

$$\tan\delta(\omega) = J''(\omega)/J'(\omega) \qquad [16]$$

at the point of consideration is small against unity, formula [9] will be rather accurate. For, by the methods to be introduced in Section 4, it will be shown that the relative error of formulae [9] will always be bounded between -0.15 ($\tan\delta$) and $+0.15$ ($\tan\delta$). Therefore, formula [9] will have a relative error smaller than 1.5% in all cases where $\tan\delta$ is smaller than 0.10.

The finite difference, $J(2t) - J(t)$, occurring in formula [9] will be approximately proportional to the derivative of the creep compliance with respect to the logarithm of time, taken at time $\sqrt{2}\,t$. Therefore we have the alternative formulation:

$$J'(\omega) \sim J(t) - 0.257\,[dJ(\xi)/d\log\xi]_{\xi=1.41t} \qquad [9']$$
$$= J(t) - 0.592\,[dJ(\xi)/d\ln\xi]_{\xi=1.41t}\,.$$

Eq. [9′] may be more familar; eq. [9] will be more useful for practical applications.

For large values of ($\tan\delta$) more involved formulae should be used. The most com-

Table 1. Numerical formulae for calculation of storage compliance from creep compliance: $J'(\omega) \sim A'(t)$; $t = 1/\omega$
$A'(t) = J(t) - a\,[J(32t) - J(16t)] - b\,[J(16t) - J(8t)] - c\,[J(8t) - J(4t)] - d\,[J(4t) - J(2t)] - e\,[J(2t) - J(t)] - f\,[J(t) - J(t/2)] - g\,[J(t/2) - J(t/4)] - h\,[J(t/4) - J(t/8)]$

a	b	c	d	e	f	g	h	bounds for relative error; %	formula number
				0.855				14.6 tan δ; — − 14.6 tan δ; —	[9]
			0.445	—	0.376			7.8 tan δ; — − 7.7 tan δ; —	[10]
		−0.0990	0.608	—	0.358			7.5 tan δ; 5.2 − 7.5 tan δ; − 9.1	[11]
		−0.119	0.680	—	0.225	—	0.0429	2.1 tan δ; 5.9 − 2.1 tan δ; − 20	[12]
	0.0108	−0.168	0.734	—	0.235			8.8 tan δ; 1.5 − 1.9 tan δ; − 1.5	[13]
	0.0109	−0.169	0.739	—	0.214	—	0.0451	2.3 tan δ; 1.6 − 2.3 tan δ; − 1.6	[14]
− 0.000715	0.0185	−0.197	0.778	—	0.181	—	0.0494	3.1 tan δ; 0.8 − 3.1 tan δ; − 0.8	[15]

plicated formula of table 1 is:

$$J'(\omega) \sim J(t) + 0.0007\,[J(32\,t) - J(16\,t)] - 0.0185\,[J(16\,t) - J(8\,t)] + 0.197\,[J(8\,t) - J(4\,t)] - 0.778\,[J(4\,t) - J(2\,t)] - 0.181\,[J(t) - J(t/2)] - 0.049\,[J(t/4) - J(t/8)] \quad [15]$$

where

$$\omega = 1/t\,.$$

We note that the difference between $J(t)$ and $J'(1/t)$ is always positive. From eq. [15], this difference is approximated by a sum of six terms of the finite difference type. The principal term in this sum is the one proportional to:

$$J(4\,t) - J(2\,t)\,.$$

At the left of the principal term are the terms which constitute the long time contribution to $J(t) - J'(1/t)$. These terms have alternating signs. The magnitude of their coefficients decreases very fast with their order. At the right of the principal term are the terms which constitute the short time contribution to $J(t) - J'(1/t)$. These terms are all positive. Their coefficients decrease also very fast with their order. Every term following the last term in eq. [15] would be shifted a factor of 4 to shorter times and would have a coefficient which is 16 times smaller than the coefficient of its predecessor.

The pyramidal structure of eq. [15] shows that the value of $J'(\omega)$ depends on the value of $J(t_0)$ at point $t_0 = 1/\omega$, and on the derivative of the creep compliance with respect to the logarithm of time, in a time interval around point t_0. The behaviour of $J(t)$ for $t > t_0$ stronger influences the calculation than does the behaviour of $J(t)$ for $t < t_0$. To apply eq. [15], the behaviour of $J(t)$ should be known in a finite time interval around t_0, viz. from $t_0/8$ to $32\,t_0$.

Eq. [15] is very accurate, whatever the value of $(\tan\delta)$ might be. It will be shown that the relative error in this formula will always be bounded between -0.8% and $+0.8\%$; moreover the relative error will also be bounded between $-3.1\ (\tan\delta)\,\%$ and $+3.1\ (\tan\delta)\,\%$. These error bounds have been indicated in the last column but one of table 1.

The way in which formula [15] was obtained is now illustrated with reference to fig. 1. We start with the intensity function of $J(t) - J'(1/t)$, which is equal to:

$$\chi(x) - \chi'(x) = 1/(1 + x^2) - \exp(-x)\,. \quad [17]$$

This intensity function is represented by the heavily drawn line in fig. 1. It is positive for all positive x-values with a maximum in the vicinity of $x = 1/2$. It increases linearly with x for small x-values and it decreases as x^{-2} for large x-values. This function is approximated by a sum of six intensity functions of the finite difference type in the following manner:

$$\chi(x) - \chi'(x) \sim \varphi'(x) = a\,\psi(x;16) + b\,\psi(x;8) + c\,\psi(x;4) + d\psi(x;2) + f\,\psi(x;1/2) + h\,\psi(x;1/8) \quad [18]$$

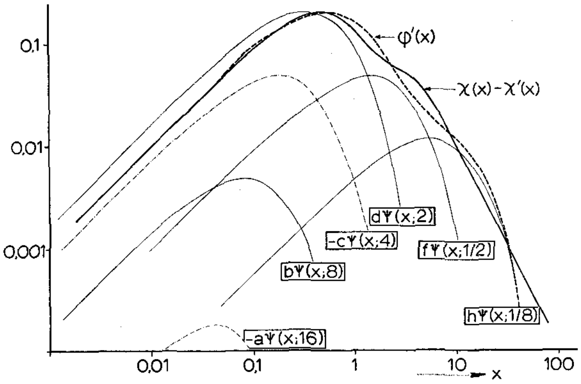

Fig. 1. Intensity function, $\chi(x) - \chi'(x)$, of difference $J(t) - J'(1/t)$, and intensity function, $\varphi'(x)$, of approximation $J(t) - A'(t)$ to this difference according to formula [15], plotted vs. x. Also shown are the intensity functions of the six finite difference terms in formula [15] which are used to construct approximation $\varphi'(x)$ (cf. eq. [18])

where $a, b, c \ldots$ are constant coefficients which have been chosen appropriately. These six terms are shown in fig. 1 together with the resulting approximation $\varphi'(x)$ [dashed line].

Approximation [18] yields an approximation for the intensity function $\chi'(x)$ by:

$$\chi'(x) \sim \chi(x) - \varphi'(x) \quad [19]$$

and, therefore, an approximation for $J'(\omega)$ by formula [15]. The relative error of approximation [19] is given by the function:

$$\Delta'(x) \equiv [\chi(x) - \varphi'(x) - \chi'(x)]/\chi'(x) \quad [20]$$

which is called the *relative error function* related to the approximation [15]. The reason is that the course of $\Delta'(x)$ vs. x determines the accuracy of the related approximation formula.

The course of function $\Delta'(x)$ vs. x is shown in fig. 2. $\Delta'(x)$ tends to zero for $x \to 0$ and for $x \to \infty$. It shows a number of deviations from zero in the intermediate x-region. The

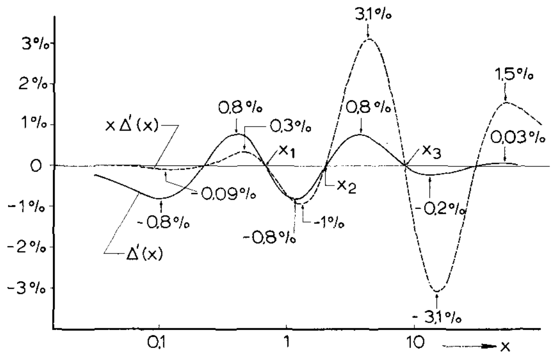

Fig. 2. Relative error function, $\Delta'(x)$, for approximation [15], vs. x; also shown is the function $x \cdot \Delta'(x)$ for the same approximation (broken line). Maximum deviations of both functions from zero have been indicated

maximum values of these deviations have been indicated. All deviations in $\Delta'(x)$ are smaller than 0.8%.

The same figure shows function $x \cdot \Delta'(x)$, which will also be of importance for the accuracy of formula [15]. This function also tends to zero for $x \to 0$ and for $x \to \infty$. Its deviations from zero remain smaller than 3.1%.

The way in which the six coefficients $a, b, c \ldots$ in formula [15] have been determined can now be explained: We imposed two sets of three conditions each on function $\Delta'(x)$ as follows:

(1) $x\,\Delta'(x)$, $\Delta'(x)$ and $\Delta'(x)/x$ should vanish at $x = 0$;

(2) $\Delta'(x)$ should vanish at the three selected x-values: $x = x_1$, $x = x_2$, $x = x_3$.

By the first set of conditions it is attained that $\Delta'(x)$ becomes a good approximation for $\chi(x) - \chi'(x)$ on the left flank of the maximum of the intensity function [i. e. between $x = 0$ and $x = 1/2$]. By the second set of conditions it is attained that $\varphi'(x)$ becomes a good approximation for $\chi(x) - \chi'(x)$ near the maximum and on the upper part of the right flank of the maximum of the intensity function.

Points x_1, x_2, x_3 are indicated in fig. 2. They were chosen by trial and error in such a way that the deviations of $\Delta'(x)$ and of $x \cdot \Delta'(x)$ from zero were minimized and that the various maxima and minima in these functions are distributed as regularly as possible. We found as a good choice: $x_1 = 0.673$, $x_2 = 1.99$, $x_3 = 8.26$.

For cases on which not enough knowledge is available to apply formula [15], simpler formulae have been listed in table 1. Formulae [13] and [14] are still applicable for very large values of tan δ. Formulae [9] to [12] should be used for small and intermediate (tan δ)-values. A more detailed comparison of the accuracies of the system of table 1 will be given in Section 4.

The way in which these formulae were derived is similar to that described above for formula [15]. However, for formulae [13] and [14] only the two conditions $x\,\Delta'(x) = 0$ and $\Delta'(x) = 0$ were imposed at $x = 0$; for [11] and [12] only one condition was imposed at $x = 0$, viz. $x\,\Delta'(x) = 0$ and for [9] and [10] $x\,\Delta'(x)$ was assumed to remain finite for $x = 0$.

3. Numerical Formulae for Calculation of Loss Compliance from Creep Compliance

Various numerical formulae for the calculation of $J''(\omega)$ from $J(t)$ are listed in

Table 2. Numerical formulae for calculation of loss compliance from creep compliance: $J''(\omega) \sim A''(t)$; $t = 1/\omega$
$A''(t) = d\,[J(4\,t) - J(2\,t)] + e\,[J(2\,t) - J(t)] + f\,[J(t) - J(t/2)] + g\,[J(t/2) - J(t/4)] + h\,[J(t/4) - J(t/8)] + j\,[J(t/8) - J(t/16)] + k\,[J(t/16) - J(t/32)] + l\,[J(t/32) - J(t/64)] + m\,[J(t/64) - J(t/128)] + n\,[J(t/128) - J(t/256)]$

d	e	f	g	h	j	k	l	m	n	bounds for relative error; %	formula number
		2.12								8 [1 + 1/(tan δ)]; 26 − 8 [1 + 1/(tan δ)]; − 26/(tan δ)	[21]
− 0.470	1.715	—	0.902							0.7 [1 + 1/(tan δ); 2.3 − 4.6/(tan δ)	[22]
− 0.505	1.807	—	0.745	—	0.158					1.1 [1 + 1/(tan δ)]; 3.5 − 1.3/(tan δ)	[23]
− 0.470	1.674	0.196	0.627	—	0.194					0.7 [1 + 1/(tan δ)]; 1.3 − 2.5 [1 + 0.5/(tan δ)]; − 2.7/(tan δ)	[24]
− 0.470	1.674	0.197	0.621	0.011	0.172	—	0.0475			0.7 [1 + 1/(tan δ)]; 2.3 − 2.5 [1 + 0.12/(tan δ)]; − 2.7/(tan δ)	[25]
− 0.470	1.674	0.198	0.620	0.012	0.172	—	0.0430	—	0.0122	0.7 [1 + 1/(tan δ)]; 2.7 − 2.5 [1 + 0.03/(tan δ)]; − 2.7/(tan(δ)	[26]
− 0.470	1.674	0.198	0.620	0.012	0.172	—	0.0433	—	0.0108	0.7 [1 + 1/(tan δ)]; 2.7 − 2.7; − 2.7/(tan δ)	[27]

table 2. The simplest of these formulae is:

$$J''(\omega) \sim 2.12\,[J(t) - J(t/2)] \text{ with } \omega = 2\,\pi\,\nu = 1/t\,. \quad [21]$$

In this formula, ω is the angular frequency and ν is the frequency at which loss compliance is calculated. The loss compliance is approximated by the finite difference between the creep compliance at times $1/\omega$ and $1/2\,\omega$. This finite difference is approximately proportional to the derivative of the creep compliance with respect to the logarithm of time, taken at the time $t/\sqrt{2}$. Therefore, we have the alternative formulation:

$$J''(\omega) \sim 0.638\,[dJ(\xi)/d\log\xi]_{\xi=0.71\,t}$$
$$= 1.47\,[dJ(\xi)/d\ln\xi]_{\xi=0.71\,t=0.113/\nu}\,. \quad [21']$$

Formula [21] is a very rough approximation. For instance, for $(\tan\delta) = 1$ the error bounds of this formula are $+\,16\%$ and $-\,16\%$. For smaller values of $\tan\delta$, the error may be even higher. Therefore, formula [21] should be used only for cases where $\tan\delta(\omega)$ is high.

A formula which is very similar to [21′] has been proposed by *Hamon* (4) for the analogous problem of calculating the imaginary part of the complex dielectric constant, ε'', from the transient dielectric response after a step in voltage. *Hamons* formula reads:

$$\varepsilon''(\nu) \sim \frac{1.59}{C_0\,V}\,[t\,i(t)]_{t=0.10/\nu} \quad [21'']$$

where $i(t)$ is the time-dependent current, C_0 the capacity of the measuring electrodes without sample and V the applied voltage. In order to translate the dielectric equation into the one for mechanical creep, we have to replace ε'' by J'' and the reduced current $i(t)/C_0\,V$ by $dJ(t)/dt$. Then *Hamons* formula changes to the form [21′] with slightly different constants, viz. 1.59 instead of 1.47 and 0.10 instead of 0.113[1]).

A much better approximation for $J''(\omega)$ can be constructed, if more than one finite difference term is used. The second best of the formulae of table 2 involves eight terms:

$$\begin{aligned}J''(\omega) \sim &-0.470\,[J(4\,t) - J(2\,t)] + 1.674\,[J(2\,t)\\ &- J(t)] + 0.198\,[J(t) - J(t/2)]\\ &+ 0.620\,[J(t/2) - J(t/4)] + 0.012\,[J(t/4)\\ &- J(t/8)] + 0.172\,[J(t/8) - J\,(t/16)]\\ &+ 0.043\,[J(t/32) - J(t/64)]\\ &+ 0.012\,[J(t/128) - J(t/256)]\end{aligned} \quad [26]$$

where

$$\omega = 1/t\,.$$

The principal term in this sum is the one proportional to

$$J(2\,t) - J(t)\,.$$

To the left of the principal term is the term which constitutes the long time contribution to $J''(1/t)$. This is the only term with negative sign. To the right of the principal term are the terms which constitute the short time contribution to $J''(1/t)$. These terms are all positive. Their coefficients decrease very slowly with their order. For instance, the term proportional to $[J(t/128) - J(t/256)]$ has a coefficient which is still $7^0/_{00}$ of the principal term. The pyramidal structure of formula [26] shows that the value of $J''(\omega)$ depends on the derivative of the creep compliance with respect to the logarithm of time in a broad time interval around point $t_0 = 1/\omega$. The behaviour of $J(t)$ for $t < t_0$ affects the calculation much stronger than does the behaviour of $J(t)$ for $t > t_0$. To actually apply formula [26], the behaviour of $J(t)$ should be known from $t_0/256$ to $4\,t_0$, i. e. from $2^1/_2$ decades to the left, to $^1/_2$ decade to the right of the point of interest. A similar conclusion had elsewhere (1, 5) already been drawn from another point of view.

It will not always be necessary to know all terms in formula [26] with high accuracy in order to be able to apply this formulae. It will often be sufficient to know upper bounds for the magnitude of the terms far away from the point of interest to justify their omission. For instance, if we know that $[J(t/32) - J(t/64)]$ and $[J(t/128) - J(t/256)]$ are not larger than $[J(2\,t) - J(t)]$, and if a 2.5% error in the calculation of J'' is admitted, the last two terms in formula [26] may be omitted.

The way in which formula [26] was derived is illustrated with reference to fig. 3. We consider the intensity function of $J''(1/t)$ which is $\chi''(x)$. This intensity function is represented by the heavily drawn line in fig. 3. It is positive for all positive x-values with a maximum at $x = 1$. It increases linearly with x for small x and decreases as x^{-1} for large x. This function is approximated by a sum of eight intensity functions of the finite difference type as follows:

[1]) Bounds for the relative error of *Hamons* formula are, in %: 51; $14\,[1 + 1/(\tan\delta)]$; $21/(\tan\delta)$ and $-\,7.5\,[1 + 1/(\tan\delta)]$; $-\,63/(\tan\delta)$.

$$\chi''(x) \sim \varphi''(x) = d\psi(x, 2) + e\,\psi(x; 1) + f\,\psi(x; 1/2) + \ldots \qquad [28]$$

These eight terms are shown in fig. 3, together with the resulting approximation

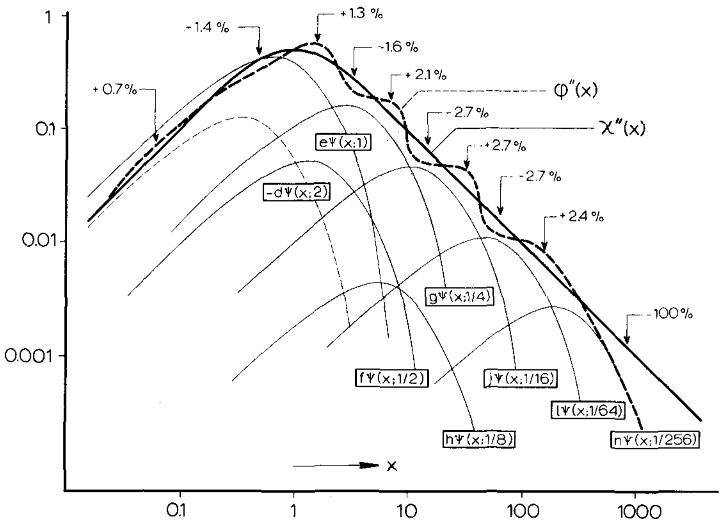

Fig. 3. Intensity function, $\chi''(x)$, of loss compliance $J''(1/t)$, and intensity function, $\varphi''(x)$, of approximation $A''(t)$ of $J''(1/t)$ according to formula [26], vs. x. Also shown are the intensity functions of the eight finite difference terms in formula [26] which are used to construct approximation $\varphi''(x)$ (cf. eq. [28]). Indicated are position and values of the maximum relative deviations of approximation $\varphi''(x)$ from $\chi''(x)$

$\varphi''(x)$, the dashed line. The relative error of the approximation [28] is given by the function:

$$\Delta''(x) \equiv [\varphi''(x) - \chi''(x)]/\chi''(x) \qquad [29]$$

which is called the relative error function related to the approximation [26]. The function $\Delta''(x)$ is zero for $x = 0$. It shows a number of deviations from zero in positive or negative direction. The maximum values of these deviations have been indicated in fig. 3. Up to an x-value of $x = 190$, all deviations remain smaller than 2.7%. At $x \sim 190$, however, the approximation falls short and $\Delta''(x)$ tends sharply to $-$ 100%. At $x = 500$, Δ'' is $-$ 25%, at $x = 1000$, Δ'' is $-$ 75%.

Formula [26] will be a very good approximation in most cases. We shall show that the relative error in this formula will always be smaller than 2.7%. The lower bound for the relative error, however, depends in a rather complicated way on the value of $\tan\delta(\omega)$ (see fig. 4). The relative error will remain above $-$ 4% in all cases where ($\tan\delta$) is in the $0.05 < \tan\delta < \infty$ region. For ($\tan\delta$)-values smaller than 0.05, the lower bound for the relative error rapidly drops. It is, e. g. $-$ 10% for $\tan\delta = 0.01$ and $-$ 75% for $\tan\delta = 0.001$. The conclusion is that formula [26] will never yield values for $J''(\omega)$ that are essentially too high; however, for small values of $\tan\delta$, it could yield results which are considerably too small.

It is possible to give a formula for $J''(\omega)$ that is applicable for *all* values of ($\tan\delta$), however small they might be. This is

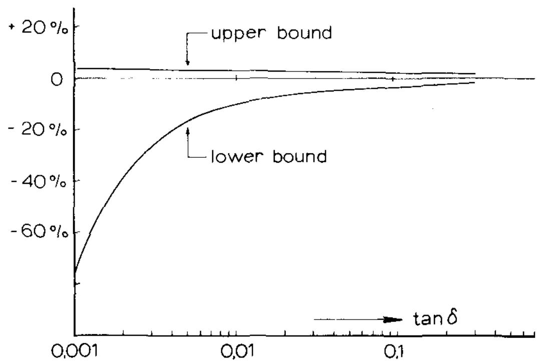

Fig. 4. Illustration of the range for the relative error in formula [26] as function of value of $\tan\delta(\omega)$

formula [27]; it is assumed to consist of an *infinite* number of terms. Each term following the one with coefficient n will be shifted a factor of 4 in time scale into the direction of shorter times relative to its predecessor, and will have a coefficient which is exactly $^1/_4$ of the coefficient of its predecessor. Instead of truncating the formula at the short time end (as was done for the other formulae of table 2), formula [27] is assumed to be an infinite series. Formula [27] will then have a relative error that is certainly bound between $-$ 2.7% and $+$ 2.7% for all values of $\tan\delta$.

In many cases not enough knowledge will be available to apply the complete formula [26] or [27]. For those cases, a number of simpler formulae has been listed in table 2. Of course, the accuracy of those formulae will be the less, the fewer terms are involved. A more detailed comparison of the accuracies of the formulae of table 2 will be given in the next section.

4. Error Bounds for Approximations

Finally, we will discuss the accuracy of the numerical formulae given in tables 1 and 2. To this end we will derive bounds for the errors of those formulae. The only assumptions made for this purpose are the ones that were already stated in the introduction: validity of the principle of superposition and the existence of a retardation spectrum which is non-negative[1]).

Let $A'(t)$ be one of the approximations for $J'(\omega)$, and $A''(t)$ one of the approximations for $J''(\omega)$. We define the errors of those approximations as the difference between the approximation and the real value of the quantity. As all approximations are linear expressions in $J(\alpha t)$ with constant coefficients, the errors may be written as integral transformations over the retardation spectrum. Using eqs. [2], [3], [4], [20] and [29], we find for the errors[2]):

$$\varepsilon'(\omega) \equiv A'(t) - J'(\omega) = \int_0^\infty f(\tau)\,\chi'(x)\,\Delta'(x)\,d\tau \quad [30]$$

$$\varepsilon''(\omega) \equiv A''(t) - J''(\omega) = \int_0^\infty f(\tau)\,\chi''(x)\,\Delta''(x)\,d\tau \quad [31]$$

where $\Delta'(x)$ and $\Delta''(x)$ are the relative error functions defined earlier; they may be obtained by inserting eqs. [1] to [8] into the corresponding definition of either table 1 or table 2. For a number of approximations, $\Delta'(x)$, $x\,\Delta'(x)$ and $\Delta''(x)$ have been plotted vs. x in figs. 5, 6 and 7.

Fig. 5 shows $\Delta'(x)$ vs. x for approximations [13], [14] and [15]. In all cases $\Delta'(x)$ vanishes for very small x and for very large x, with a number of maxima and minima in the region of intermediate x-values. Therefore, $\Delta'(x)$ is bounded for all positive x-values by

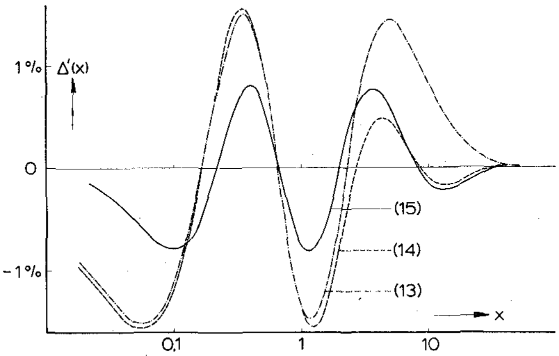

Fig. 5. The course of relative error functions, $\Delta'(x)$, for approximations [13], [14] and [15], vs. x

a positive upper bound and a negative lower bound. The absolute values of both bounds are small compared with unity.

Comparing expression [30] with the definition of $J'(\omega)$ in the following form:

$$J'(\omega) = J_0 + \int_0^\infty f(\tau)\,\chi'(x)\,d\tau \quad [2]$$

we can immediately bound error $\varepsilon'(\omega)$ in terms of a small positive or negative fraction of $J'(\omega)$:

$$J'(\omega)\,\{\Delta'(x)\}_{\min} \le \varepsilon'(\omega) \le J'(\omega)\,\{\Delta'(x)\}_{\max}\,. \quad [32]$$

This leads to a small positive upper bound and a small negative lower bound for the relative error $\varepsilon'(\omega)/J'(\omega)$ [cf. column 9 of table 1].

Fig. 6 shows the course of $x\,\Delta'(x)$ vs. x for approximations [9], [10], [11] and [12]. It is seen from this figure that also function $x\,\Delta'(x)$ is bounded for all positive x-values between a positive upper bound and a negative lower bound.

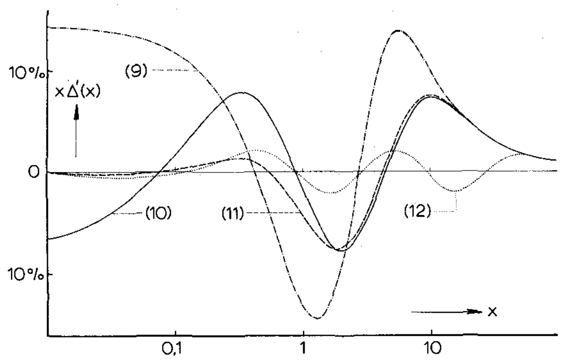

Fig. 6. The course of functions $x \cdot \Delta'(x)$ for approximations [9], [10], [11] and [12], vs. x

[1]) A similar manner of deriving error bounds for approximations has been introduced in (1).

[2]) The terms in ε' and ε'', which would arise from J_0 or $1/\omega\,\eta$, cancel out in most approximations. In the few cases where such a term remains, it is without consequences for the argument.

Comparing expression [30] with the definition of $J''(\omega)$ in the following form:

$$J''(\omega) = \frac{1}{\omega\,\eta} + \int_0^\infty f(\tau)\,[\chi'(x)/x]\,d\tau \qquad [3]$$

we can bound error $\varepsilon'(\omega)$ in terms of a small fraction of $J''(\omega) = J'(\omega)\tan\delta$:

$$J'(\omega)\,(\tan\delta)\,\{x\,\Delta'(x)\}_{\min} \le \varepsilon'(\omega) \le J'(\omega)\,(\tan\delta) \times \{x\,\Delta'(x)\}_{\max}\,. \qquad [33]$$

This leads to a positive and a negative bound expressed in (tan δ), as given in column 9 of table 1.

Fig. 7 shows the course of $\Delta''(x)$ vs. x for approximations [22], [24], [25], [26] and [27]. $\Delta''(x)$ vanishes for very small values of x; it shows a number of maxima and minima in the region of intermediate x-values. However, for each formula[1]), $\Delta''(x)$ finally tends to -1 for large x-values. The x-value where this transition takes place, depends on the length of the short time-tail of the formula. Consequently, $\Delta''(x)$ is bounded by a small positive upper bound. However, it does not have a useful lower bound[1]). It is therefore not possible to apply the reasoning used above to bound relative error $\varepsilon''(\omega)/J''(\omega)$.

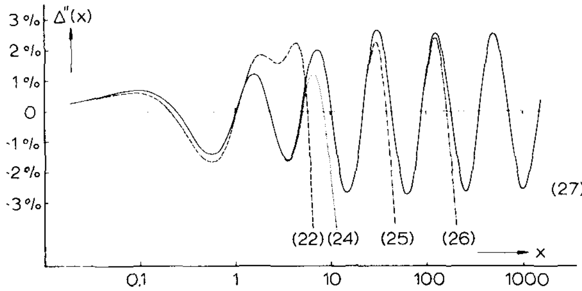

Fig. 7. The course of relative error functions, $\Delta''(x)$, for approximations [22], [24], [25], [26] and [27], vs. x

To find effective bounds for both, $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$, the integral representations of the errors are rewritten in a slightly different way:

$$\varepsilon'(\omega) = \int_0^\infty f(\tau)\,[\chi'(x) + p\,\chi''(x)]\left\{\frac{x\,\Delta'(x)}{x+p}\right\}d\tau \qquad [34]$$

$$\varepsilon''(\omega) = \int_0^\infty f(\tau)\,[\chi''(x) + q\,\chi'(x)]\left\{\frac{\Delta''(x)}{1+q\,x}\right\}d\tau \qquad [35]$$

where p and q are assumed to be two arbitrary, but non-negative numbers. Formulae [34] and [35] are identities with expressions [30] and [31]. We compare the errors with the integral representation of the following linear combinations of $J'(\omega)$ and $J''(\omega)$:

$$J'(\omega) + p\,J''(\omega) = J'(\omega)\,[1 + p\,(\tan\delta)] = \int_0^\infty f(\tau)\,[\chi'(x) + p\,\chi''(x)]\,d\tau + J_0 + \frac{p}{\omega\,\eta} \qquad [36]$$

$$J''(\omega) + qJ'(\omega) = J''(\omega)\,[1 + q/(\tan\delta)] = \int_0^\infty f(\tau)\,[\chi''(x) + q\,\chi'(x)]\,d\tau + q\,J_0 + \frac{1}{\omega\,\eta}\,. \qquad [37]$$

The terms outside the integrals in [36] and [37] are positive and of no consequence for what follows.

The only difference between the integral representations of [34] and [36] on the one hand, and of [35] and [37] on the other are the terms within brackets in ε' and ε''. For positive values of x, these terms have an upper and a lower bound, which will depend on the chosen fixed value of p or q; therefore we may write:

$$-\zeta'(p) \le \{x\,\Delta'(x)/[x+p]\} \le \xi'(p) \qquad [38]$$

for all $x \ge 0$

$$-\zeta''(q) \le \{\Delta''(x)/[1+q\,x]\} \le \xi''(q)\,. \qquad [39]$$

Though we did not find a useful lower bound of $\Delta''(x)$ itself, the absolute value of lower bound $-\zeta''(q)$ will be small, whenever the value of parameter q is chosen sufficiently large.

Because of our assumption that the retardation spectrum is non-negative, the integrands in eqs. [34], [35], [36] and [37] are positive or zero for all positive values of τ respectively x. Therefore, inequality [38] together with eqs. [34] and [36] yield the following bounds for the relative error:

$$-\zeta'(p)\,[1 + p\,(\tan\delta)] \le \varepsilon'(\omega)/J'(\omega) \le \xi'(p)\,[1 + p\,(\tan\delta)]\,. \qquad [40]$$

In a similar way we find:

$$-\zeta''(q)\,[1 + q/(\tan\delta)] \le \varepsilon''(\omega)/J''(\omega) \le \xi''(q)\,[1 + q/(\tan\delta)]\,. \qquad [41]$$

The bounds for relative error ε'/J' depend on the chosen fixed value for p and are increasing functions of the value of (tan δ) at angular frequency ω. If we repeat the argument for a different value of p, we obtain different functions of (tan δ) as bounds. If the procedure is performed for a whole sequence of p-values between zero and infinite, two families of curves result. The curves of the one family all constitute upper bounds for ε'/J', the curves of the

[1]) Except for formula [27].

other family all constitute lower bounds. The envelopes of both the families constitute most restrictive upper and lower bounds.

This procedure has been performed on a digital computer for all numerical formulae listed in tables 1 and 2. Upper and lower bounds for the relative errors in those formulae are given, as functions of (tan δ), in tables 3 to 6. Tables 3 and 4 list upper and lower bounds for formulae of the $A'(t)$ type; in tables 5 and 6 upper and lower bounds are listed for formulae of the $A''(t)$ type. Note that lower bounds, which are always negative, are listed with their absolute values only. Illustrations are given in figs. 8, 9, 10 and 11.

Table 3. Upper bounds for relative error, in %, of formulae for calculation of $J'(\omega)$ from $J(t)$, as functions of tan $\delta(\omega)$

formula / tan δ	[9]	[10]	[11]	[12]	[13]	[14]	[15]
0.1	1.5	0.78	0.75		0.9		
0.15	2.2	1.2	0.92		1.3		
0.2	2.9	1.6	1.0		1.5		0.6
0.3	4.4	2.3	1.2	0.6	1.5		0.8
0.4	5.8	3.1	1.4	0.8	1.5	0.6	0.8
0.5	7.3	3.9	1.6	1.0	1.5	0.6	0.8
0.6	8.8	4.7	1.7	1.3	1.5	0.7	0.8
0.7	10.2	5.5	1.8	1.5	1.5	0.7	0.8
0.8	11.7	6.2	1.9	1.7	1.5	0.8	0.8
1.0	14.6	7.8	2.1	2.1	1.5	0.9	0.8
1.5		11.7	2.7	3.2	1.5	1.1	0.8
2.0		15.6	3.2	4.2	1.5	1.3	0.8
3.0			4.3	5.9	1.5	1.6	0.8
4.0			5.2	5.9	1.5	1.6	0.8
5.0			5.2	5.9	1.5	1.6	0.8
6.0			5.2	5.9	1.5	1.6	0.8
7.0			5.2	5.9	1.5	1.6	0.8
8.0			5.2	5.9	1.5	1.6	0.8
10.0			5.2	5.9	1.5	1.6	0.8

Table 4. Absolute values of lower bound for relative error, in %, of formulae for calculation of $J'(\omega)$ from $J(t)$, as functions of tan $\delta(\omega)$

formula / tan δ	[9]	[10]	[11]	[12]	[13]	[14]	[15]
0.1	1.5	0.77	0.75				
0.15	2.2	1.2	1.1				
0.2	2.9	1.5	1.5				
0.3	4.4	2.3	2.3	0.6	0.6	0.7	0.4
0.4	5.8	3.1	3.0	0.8	0.8	0.9	0.5
0.5	7.3	3.9	3.8	1.0	1.0	1.2	0.6
0.6	8.8	4.6	4.5	1.3	1.1	1.3	0.6
0.7	10.2	5.4	4.9	1.4	1.3	1.5	0.7
0.8	11.7	6.2	5.0	1.5	1.5	1.6	0.8
1.0	14.6	7.7	5.0	1.7	1.5	1.6	0.8
1.5		11.6	5.1	1.9	1.5	1.6	0.8
2.0		15.4	5.1	2.1	1.5	1.6	0.8
3.0			5.1	2.5	1.5	1.6	0.8
4.0			5.2	2.9	1.5	1.6	0.8
5.0			5.2	3.3	1.5	1.6	0.8
6.0			5.3	3.7	1.5	1.6	0.8
7.0			5.3	4.1	1.5	1.6	0.8
8.0			5.4	4.5	1.5	1.6	0.8
10.0			5.5	5.2	1.5	1.6	0.8

Table 5. Upper bounds for relative error, in %, of formulae for calculation of $J''(\omega)$ from $J(t)$, as functions of tan $\delta(\omega)$

formula / tan δ	[21]	[22]	[23]	[24]	[25]	[26]	[27]
0.01	25.7	2.3	3.5	1.3	2.3	2.7	2.7
0.015	25.7	2.3	3.5	1.3	2.3	2.7	2.7
0.02	25.7	2.3	3.5	1.3	2.3	2.7	2.7
0.03	25.7	2.3	3.5	1.3	2.3	2.7	2.7
0.04	25.7	2.3	3.5	1.3	2.3	2.6	2.6
0.05	25.7	2.3	3.5	1.3	2.3	2.5	2.5
0.06	25.7	2.3	3.5	1.3	2.2	2.4	2.4
0.07	25.7	2.3	3.5	1.3	2.2	2.4	2.4
0.08	25.7	2.3	3.5	1.3	2.2	2.3	2.3
0.1	25.7	2.3	3.5	1.3	2.1	2.2	2.2
0.15	25.7	2.3	3.5	1.3	2.0	2.0	2.0
0.2	25.7	2.3	3.0	1.3	1.8	1.8	1.8
0.3	25.7	2.2	2.4	1.3	1.6	1.6	1.6
0.4	25.7	2.0	2.1	1.3	1.5	1.5	1.5
0.5	23.4	1.9	1.9	1.3	1.4	1.4	1.4
0.6	20.5	1.8	1.8	1.3	1.3	1.3	1.3
0.7	18.5	1.7	1.7	1.2	1.2	1.2	1.2
0.8	16.9	1.6	1.6	1.2	1.2	1.2	1.2
1.0	14.8	1.4	1.6	1.1	1.1	1.1	1.1
1.5	11.9	1.1	1.5	1.0	1.0	1.0	1.0
2.0	10.5	1.0	1.4				
3.0	9.0		1.3				
4.0	8.3		1.3				
5.0	7.9		1.3				
6.0	7.6		1.2				
7.0	7.4		1.2				
8.0	7.2		1.2				
10.0	7.0		1.1				

In figs. 8 and 9, upper and lower bounds for the relative error of the formulae of table 1 are shown as functions of (tan δ), in double logarithmic diagrams. The absolute value of all bounds increases with increasing value of (tan δ). Consequently, the calculation of $J'(\omega)$ from $J(t)$ will be the more difficult, the higher the value of (tan δ). From figs. 8 and 9 it will be easy to determine the appropriate approximation to be used in each particular case. If, for instance, experimental accuracy admits calculation of $J'(\omega)$ within an error of 1%, formula [9] could be used for values of (tan δ) between 0 and 0.07; formula [10] between 0 and 0.13, formula [12] between 0 and 0.5, and finally formula [15] for all (tan δ)-values.

In figs. 10 and 11 upper and lower bounds for the relative error of the formulae of table 2 are shown as functions of (tan δ), in double logarithmic diagrams. The absolute value of all bounds decreases with increasing value of (tan δ). Consequently, the calcula-

Table 6. Absolute values of lower bounds for relative error, in %, of formulae for calculation of $J''(\omega)$ from $J(t)$, as functions of tan $\delta(\omega)$

formula tan δ	[21]	[22]	[23]	[24]	[25]	[26]	[27]
0.001						76.0	2.7
0.0015						51.5	2.7
0.002						39.3	2.7
0.003					91.7	27.0	2.7
0.004					76.6	20.8	2.7
0.005					62.0	17.2	2.7
0.006					52.2	14.7	2.7
0.007					45.1	12.9	2.7
0.008					39.8	11.6	2.7
0.01					32.5	9.8	2.7
0.015			84.1	80.5	22.6	7.4	2.7
0.02			66.8	61.2	17.7	6.1	2.7
0.03			44.5	41.2	12.8	4.9	2.7
0.04			33.4	31.2	10.1	4.3	2.7
0.05		88.2	26.7	25.2	8.3	3.9	2.7
0.06		76.9	22.3	21.2	7.1	3.5	2.7
0.07		66.0	19.1	18.4	6.3	3.2	2.6
0.08		57.7	16.7	16.2	5.6	2.9	2.5
0.1	85.7	46.2	13.4	13.2	4.7	2.6	2.4
0.15	59.9	30.8	8.9	9.2	3.6	2.2	2.2
0.2	46.3	23.1	6.7	7.2	3.0	2.0	2.0
0.3	32.8	15.4	4.5	5.2	2.4	1.7	1.7
0.4	26.0	11.6	3.4	4.2	2.1	1.6	1.6
0.5	22.0	9.3	2.7	3.6	1.9	1.6	1.6
0.6	19.3	7.7	2.2	3.2	1.8	1.5	1.5
0.7	17.4	6.6	1.9	2.8	1.7	1.5	1.5
0.8	15.9	5.8	1.7	2.6	1.6	1.5	1.5
1.0	13.9	4.6	1.3	2.1	1.5	1.4	1.4
1.5	11.1	3.1	0.9	1.6	1.4	1.4	1.4
2.0	9.7	2.3		1.3	1.3	1.3	1.3
3.0	8.2	1.5		0.9	0.9	0.9	0.9
4.0	6.5	1.1					
5.0	5.2	0.9					
6.0	4.4						
7.0	3.7						
8.0	3.3						
10.0	2.6						

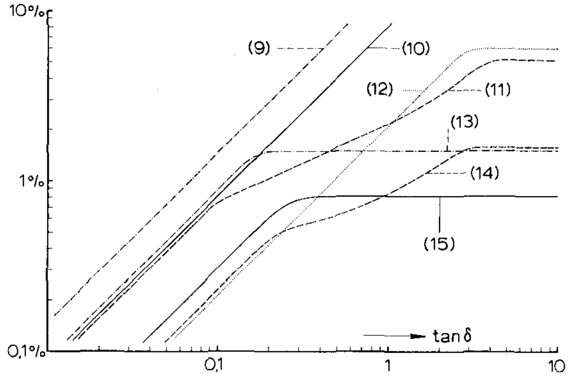

Fig. 9. Lower bounds for relative error of formulae of table 1, as functions of value of tan $\delta(\omega)$

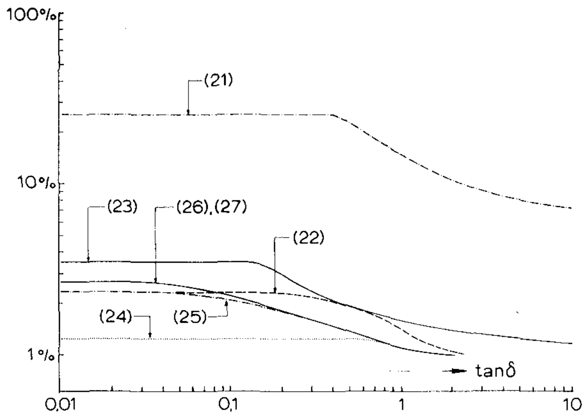

Fig. 10. Upper bounds for relative error of formulae of table 2, as functions of value of tan $\delta(\omega)$

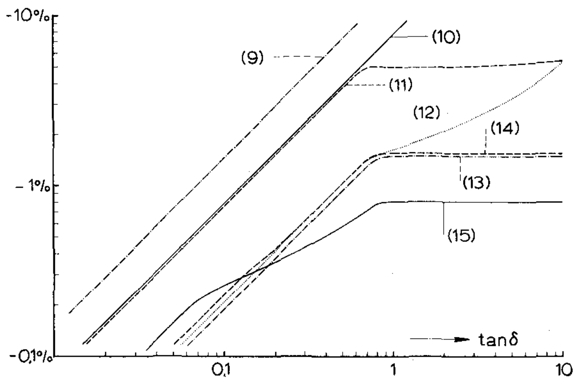

Fig. 8. Upper bounds for relative error of formulae of table 1, as functions of value of tan $\delta(\omega)$

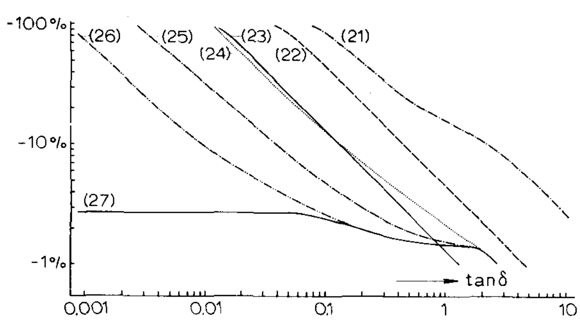

Fig. 11. Lower bounds for relative error of formulae of table 2, as functions of value of tan $\delta(\omega)$

tion of $J''(\omega)$ from $J(t)$ is the more difficult, the lower the value of $(\tan \delta)$. Consider first the upper bound for the relative error as shown in fig. 10. Only formula [21] could yield results which are considerably too high, namely 26% in the worst case. The other formulae will never yield results which are essentially higher than the real value of $J''(\omega)$. The relative error always remains below 3.5% for formula [23], 1.3% for formula [24], 2.7% for formula [27] etc. The difficulty originates from the lower bound for the relative error as shown in fig. 11. Only formula [27] may be safely used for all values of $\tan \delta(\omega)$. The other formulae have a $(\tan \delta)$-region where they may be safely used, and a $(\tan \delta)$-region where they might yield results which are considerably too low. If, for instance, experimental accuracy admits calculation of $J''(\omega)$ with an error smaller than 3%, formula [21] should not be used; formula [22] could be safely used for values of $(\tan \delta)$ between 1.5 and ∞; formula [23] between 0.45 and ∞; formula [24] between 0.65 and ∞; formula [25] between 0.2 and ∞; formula [26] between 0.075 and ∞; and formula [27] for all values of $(\tan \delta)$.

5. Concluding Remarks

We would like to emphasize that the formulae given in tables 1 and 2 will be much more accurate in most practical cases than one might conclude from the error bounds which have been derived. The error of a formula will really attain its bound in the most unfavourable situation only, viz. when the retardation spectrum consists of one sharp line which is situated at the most unfavourable place. When dealing with a smooth spectrum of retardation times, the positive and negative contributions of the error function under the integral will cancel out for the largest part. In those cases, the real error may well be one order of magnitude smaller than the bounds. This remark should apply especially to the lower bounds of the errors of the truncated formulae of table 2.

Whenever possible, one should start the considerations with the complete formula (formula [15] for $J'(\omega)$ and [27] for $J''(\omega)$). Then, by using experimental evidence on the magnitude of the logarithmic derivative of the creep compliance, one should leave out all terms with a contribution smaller than the experimental error. Often one will then end up with a formula which only involves a small number of significant terms.

A different procedure is to derive more restrictive lower bounds for the relative error of the truncated formulae of table 2, by using experimental evidence on the magnitude of the short time creep behaviour. Consider, for instance, formulae [24] and denote the error of this formula by $\varepsilon_{[24]}''$. It is possible to express this error in terms of the error of formula [27], denoted here by $\varepsilon_{[27]}''$, and the finite differences of the creep compliance in the short time domain. By comparing the definitions of [24] and [27] and by using eq. [31] twice, we obtain:

$$\begin{aligned}\varepsilon''_{[24]} = \varepsilon''_{[27]} &- 0.002\,[J(t) - J(t/2)] + 0.007\,[J(t/2) \\ &- J(t/4)] - 0.012\,[J(t/4) - J(t/8)] + 0.022\,[J(t/8) \\ &- J(t/16)] - 0.043\,[J(t/32) - J(t/64)] \\ &- 0.0108\,[J(t/128) - J(t/256)] - \ldots \end{aligned} \qquad [42]$$

We know bounds for the error ε'' [27], viz.:

$$-0.027\,J''(\omega) \leq \varepsilon''_{[27]} \leq +0.027\,J''(\omega)\,. \qquad [43]$$

From the existence of the upper bound for formula [21] we derive the following inequality:

$$J''(\omega) \geq \frac{2.12}{1.26}\,[J(t) - J(t/2)] = 1.68\,[J(t) - J(t/2)]. \qquad [44]$$

If it is possible to give bounds of the difference terms on the right hand side of eq. [42] in terms of $[J(t) - J(t/2)]$, we can immediately, by using [43] and [44], derive more restrictive lower bounds for error ε'' [24].

Consider, as an example, the frequently occurring case that the creep compliance is a convex function of the logarithm of time, i. e. that the following inequalities are true:

$$\begin{aligned} J(t/128) - J(t/256) &\leq J(t/32) - J(t/64) \leq J(t/8) \\ &- J(t/16) \leq J(t/4) - J(t/8) \leq J(t/2) \\ &- J(t/4) \leq J(t) - J(t/2)\,. \end{aligned} \qquad [45]$$

Then, by combining eqs. [42], [43], [44] and [45] we obtain the following lower bound for $\varepsilon''_{[24]}$:

$$\begin{aligned}\varepsilon''_{[24]} &\geq -0.027\,J''(\omega) - 0.042\,[J(t) - J(t/2)] \\ &\geqq -0.027\,J''(\omega) - 0.025\,J''(\omega) \\ &= -0.052\,J''(\omega)\,. \end{aligned} \qquad [46]$$

In this manner we obtained the error bounds listed in table 7.

A discussion on applications of the formulae proposed here is postponed to a following publication.

Table 7. Bounds for the relative error, in %, for formulae of table 2 under the condition of convex creep behaviour

formula number	upper bounds	lower bounds
[22]	2.3	− 9.4
[24]	1.3	− 5.2
[25]	2.3	− 3.4
[26]	2.7	− 2.9

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Summary

Numerical formulae are given for calculation of storage and loss compliance from the course of the creep compliance for linear viscoelastic materials. These formulae involve values of the creep compliance at times which are equally spaced on a logarithmic time scale. The ratio between succeeding times corresponds to a factor of two.

A method is introduced by which bounds for the relative error of those formulae can be derived. These bounds depend on the value of the damping, tan δ. The calculation of the storage compliance is easier with the lower damping values. This calculation involves the value of the creep compliance at time $t_0 = 1/\omega$, and that of its derivative with respect to the logarithm of time in a rather narrow region around t_0. In contrast the calculation of the loss compliance is more difficult with the lower damping values. This calculation involves the value of the derivative of the creep compliance with respect to the logarithm of time in a broad interval around t_0.

Zusammenfassung

Numerische Formeln werden angegeben, die die Berechnung der dynamischen Nachgiebigkeit aus der Kriechkurve ermöglichen. In diesen Formeln treten Werte der Kriechkurve auf, die zu logarithmisch äquidistanten Zeitpunkten gemessen wurden. Das Verhältnis zweier aufeinanderfolgender Zeitpunkte entspricht stets einem Faktor 2.

Für alle Formeln werden obere und untere Schranken für den relativen Fehler abgeleitet. Diese Schranken hängen vom Werte der Dämpfung (tan δ) ab, die bei der Kreisfrequenz ω auftritt, für die die Berechnung erfolgt. Die Berechnung der Speicherkomponente der dynamischen Nachgiebigkeit ist desto leichter, je niedriger der Wert der Dämpfung ist. Zu dieser Berechnung benötigt man den Wert der Kriechfunktion zum Zeitpunkt $t_0 = 1/\omega$ und deren logarithmische Zeitableitung in einem ziemlich engen Zeitintervall um t_0. Die Berechnung der Verlustkomponente der dynamischen Nachgiebigkeit ist desto leichter, je höher der Wert der Dämpfung ist. Zu dieser Berechnung benötigt man den Wert der logarithmischen Zeitableitung der Kriechfunktion in einem breiten Zeitintervall um t_0.

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V-t Capillary Rheometry

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With 3 figures in 4 details and 1 table

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List of Symbols

D sec^{-1})	rate of shear
g (cm · sec^{-2})	acceleration of gravity
h (cm)	difference in heights in liquid level
H (cm)	maximum value of h
L_0 (cm)	calibrated length of burette
L_c (cm)	length of capillary
ΔP (dyn · cm^{-2})	pressure drop
Q (cm^3 · sec^{-1})	volumetric flow rate
R_0 (cm)	internal radius of burette
R_c (cm)	internal radius of capillary tube
t (sec)	time of flow
V (cm^3)	volume of liquid flown out of the instrument
τ (dyn · cm^{-2})	shear stress
ϱ (g · cm^{-3})	density of liquid
$\mathfrak{F}$ (sec^{-1})	kinematic consistency variable
$\mathfrak{P}$ (dyn · cm^{-2})	dynamic consistency variable
A (1)	dimensionless dynamic consistency variable
B (1)	dimensionless kinematic consistency variable
p (1)	dimensionless rate of shear
T (1)	dimensionless time
y (1)	dimensionless pressure drop
ϑ (1)	dimensionless shear stress